Berkeley mini Math Tournament 2024

Team Round

April 14, 2024

Time limit: 60 minutes.

Instructions: For this test, you will work in teams of up to five to solve 20 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Submit a single answer sheet for grading. Only answers written inside the boxes on the answer sheet will be considered for grading.

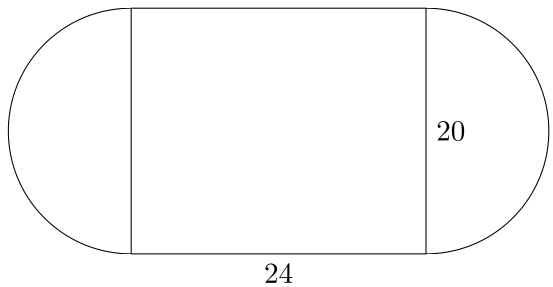
No calculators.

Answer format overview:

- Carry out any reasonable calculations. For instance, you should evaluate $\frac{1}{2}+\frac{1}{3}$, but you do not need to evaluate large powers such as 7^8.
- Write rational numbers in lowest terms. Decimals are also acceptable, provided they are exact. You may use constants such as π in your answers.
- Move all square factors outside radicals. For example, write $3\sqrt{7}$ instead of $\sqrt{63}$.
- Denominators do *not* need to be rationalized. Both $\frac{\sqrt{2}}{2}$ and $\frac{1}{\sqrt{2}}$ are acceptable.
- Do not express an answer using a repeated sum or product.

1. Find the positive integer x such that $x^x - \frac{12}{x^x} = 1$.

2. A track is constructed by attaching two semicircular arcs with diameter 20 meters to a rectangle with width 20 meters and length 24 meters, as shown below. Preston runs a lap around the entire track, while Ronan only runs a lap around the rectangle. What is the absolute difference between the distance Preston runs and the distance Ronan runs, in meters?



3. What is the sum of the odd factors of 2024?

4. Oski the bear wears a sock on each of his four distinguishable paws, where each paw has either a white sock or a black sock. If his four socks are not all the same color, find the number of different sock combinations Oski can wear.

5. Eight friends are meeting up to share books with each other. Each friend brings some number of books, and the median number of books that the friends brought is 11. If six of the friends bring $2, 10, 12, 16, 20$, and 24 books, respectively, what is the maximum number of books the other two friends could have brought in total?

6. Austin writes the integers from 1 to 1000, inclusive, on a blackboard. He then erases every integer that is divisible by 5. How many instances of the digit 5 are left on the blackboard?

7. Kiran has two buckets, Bucket A and Bucket B, each with lots of water, and a hose that can add water at a fixed rate to the buckets. Kiran punctures a hole into Bucket A so it leaks out water at a constant rate into Bucket B. If Kiran uses the hose to add water to Bucket A, Bucket A will first be holding 1 more gallon of water 48 seconds later. If he instead uses the hose to directly add water to Bucket B, Bucket B will first be holding 1 more gallon of water 16 seconds later. How many seconds does it take for a gallon of water to leak from Bucket A into Bucket B?

8. Mary has a box that is 7 inches long and wide and 4 inches tall. What is the maximum number of 2 inch by 2 inch by 1 inch blocks she can fit inside the box without it overflowing?

9. A deck of eight distinct cards is shuffled in a particular way. In one round of shuffling, the following is done in order:

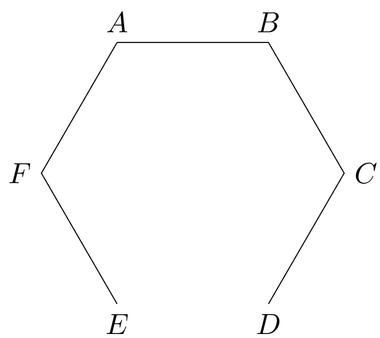
   - The second card from the top is moved to the top.
   - The third card from the top is moved to the top.
   - The fifth card from the top is moved to the top.
   - The seventh card from the top is moved to the top.

   What is the least number of rounds of shuffling that would return the deck to its original order?

10. Recall that the arithmetic mean of n numbers $a_1, \ldots, a_n$ is $\frac{a_1+\cdots+a_n}{n}$ and the geometric mean of n numbers $a_1, \ldots, a_n$ is $\sqrt[n]{a_1 \ldots a_n}$. A five-element arithmetic sequence a_1, a_2, a_3, a_4, a_5 has a common difference of y and an arithmetic mean of x. A five-element geometric sequence b_1, b_2, b_3, b_4, b_5 has a common ratio of y and a geometric mean of x. If $a_1 = b_1 = 2024$, compute the sum of all possible values of y.

11. Jessica has an analog clock on her wall. At a certain time, she measures the smaller angle between the hour and minute hand and finds that it is 138°. She also has a digital clock on her nightstand showing the same time, and the product of the number of minutes and hours displayed is 72. What is the least number of minutes that could have passed since 12:00 A.M. of the same day?

12. A dog is on a leash that is attached to the exterior of a fenced region in the shape of a regular hexagon, $ABCDEF$, with side length 1 foot. The dog's leash is 3 feet long and is attached to vertex A. If side $\overline{DE}$ is removed, as shown below, what is the area of the region the dog can reach while still attached to the leash?



13. Compute the sum of all values of c for which there exists a function f such that for all real numbers a and b, the following hold:

$$f(a) + f(a^2) = c,$$
$$f(0) + f(1-b) + f(1-2b+b^2) = c + 20.$$

14. Five people, Alice, Benji, Carl, Danielle, and Erica, are passing around a bag with 6 candies in it. Alice initially holds the bag and takes a candy from the bag. Then, she passes the bag and remaining candies to one of the other four people uniformly at random. That person takes a candy from the bag and then passes the bag and remaining candies to one of the other four people uniformly at random. This continues until there are no more candies in the bag. What is the probability that every person takes a candy from the bag?

15. Let S be the sum of all positive integers $n \leq 91$ such that $n^2 + 2n + 18$ is divisible by 91. Compute the remainder when S is divided by 91.

16. Triangle $\triangle ABC$ has point D on side $\overline{AB}$ so that $AD = BD$ and $BC = CD$. Point E is placed on side $\overline{AC}$ so that $\angle ABC = 2\angle ADE$. If $AE = 4$ and $CE = 12$, find BC.

17. Tushar has three fair 20-sided dice, D_1, D_2, and D_3. Die D_1 has faces labeled with the integers from 1 to 20, inclusive, die D_2 has faces labeled with the even integers from 2 to 40, inclusive,

and die D_3 has faces labeled with the multiples of 3 from 3 to 60, inclusive. Tushar selects one of these three dice from a bag uniformly at random and then rolls the selected die three times. If the sum of Tushar's three rolls is 30, what is the probability that he selected D_1?

18. Let p, q, and r be the three roots of the cubic polynomial $x^3 - 5x + 1$. Compute
$$\frac{1}{p^3+5} + \frac{1}{q^3+5} + \frac{1}{r^3+5}.$$

19. A right cylinder is placed in a cube with side length 1 such that it is tangent to all 6 faces of the cube and its altitude is parallel to a main diagonal of the cube. What is the height of the cylinder, given that it is twice the cylinder's diameter?

20. Shreyas is playing a game. He begins with 2 marbles in his collection. Then, on turn n of the game, he flips a fair coin and adds 2^n marbles to his collection if it lands heads. The game ends when the number of marbles in his collection is divisible by either 3 or 5. What is the probability that the number of marbles in his collection is divisible by 5 at the end of the game?