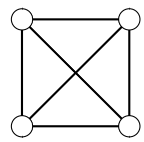
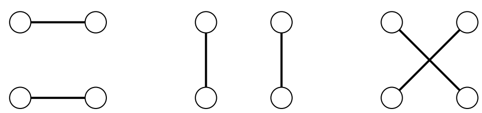
1. Introduction

We begin by introducing some terminology about graphs. A (simple) *graph* is comprised of a set of *vertices* V together with a set of *edges* E, which are two-element subsets of V. Define the *degree* of a vertex v as the number of edges in E that contain v, and the number of vertices in G to be its *order*. A *bipartite* graph is one where the vertices can be split into two sets such that no edge appears between vertices of the same set. Define a *cycle* to be a set of vertices $v_1, ..., v_n$ such that each v_i and v_{i+1} has an edge between them, as does a_n and a_1. Define a *perfect matching* to be a set of edges E' in G where each vertex is the endpoint of exactly one edge in E'; for example, in the graph

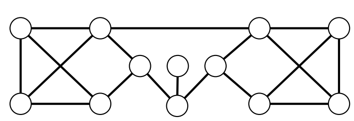


we have the following three perfect matchings:



2. Matchings

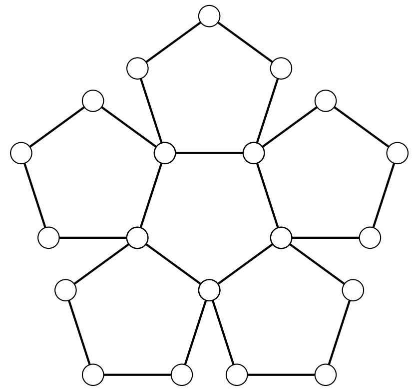
1. (a) **[1]** Draw a perfect matching on the following graph.



We define K_n to be the complete graph on n vertices; that is, we have n vertices with each pair of distinct vertices being connected by an edge.

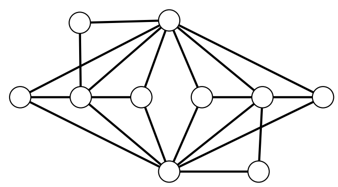
(b) **[2]** Find an expression for the number of perfect matchings of K_{2n}, and compute this value for $n = 6$.

(c) **[3]** Let P_n be a regular n sided polygon with n vertices and n edges. Let Q_n be a graph composed of $n+1$ copies of P_n, called $\mathfrak{P}_0, \mathfrak{P}_1, ..., \mathfrak{P}_n$, in the following way: Let the vertices of $\mathfrak{P}_0$ be $v_1, ..., v_n$. Then for $1 \leq k \leq n$, $\mathfrak{P}_k$ shares exactly v_k, v_{k+1} with $\mathfrak{P}_0$ (taking $v_{n+1} = v_1$), and does not share any edges with any other $\mathfrak{P}_i$. An example for $n = 5$ is given below:

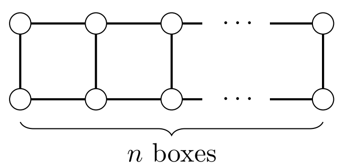


Find an expression for the number of perfect matchings of Q_n.

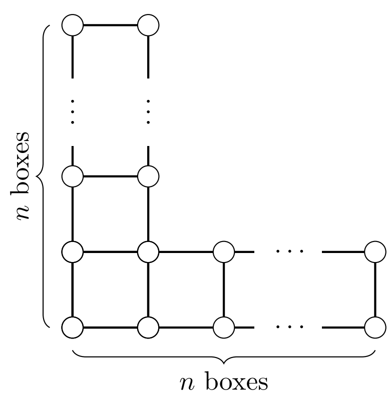
(d) **[3]** Is there a perfect matching of this graph? If so, find one. If not, prove that none exists.



2. (a) **[3]** What is the number of perfect matchings on the following graph?



(b) **[3]** What is the number of perfect matchings on this graph?



3. **[2]** A *forest* is defined as a graph having no cycles. Show that a forest has at most 1 perfect matching.

4. **[5]** Show that if G is a graph of order $2n$ so that every vertex of G has degree $\geq n$, then G has a perfect matching.

5. **[8]** Define adj(S) as the set of vertices that are adjacent to at least one vertex in S. Define G to be a bipartite graph with vertex sets V_1, V_2 (that is, all edges have endpoints in a vertex in V_1 and V_2). Prove that a bipartite graph has a perfect matching if and only if for every subset $S \subseteq V_1$, $|\text{adj}(S)| \geq |S|$ where $|S|$ denotes the size of the set S.

3. Recurrences

A *recurrence* is a sequence a_n where each new term is generated by a function of the ones before it. Often, *initial conditions* are specified, to give the starting point for the recurrence. For example, a particularly famous recurrence is the Fibonacci sequence, which has initial conditions $F_1 = 1, F_2 = 1$, and recursion formula $F_n = F_{n-1} + F_{n-2}$ for $n > 2$.

6. (a) **[3]** Show that all terms x_n of the recurrence given by $x_n x_{n-2} = x_{n-1}^2 + 1$ with initial conditions $x_0 = 1$, $x_1 = 1$ are integers.

   (b) **[4]** Find and prove an expression for the x_n in part (a) in terms of the Fibonacci numbers.

We consider sequences a_n given by recurrences of the form

$$a_n a_{n-m} = a_{n-i}a_{n-j} + a_{n-k}a_{n-l}, \qquad \text{where } m = i + j = k + l$$

and with initial conditions $a_0, a_1, \ldots, a_{m-1}$. We call this the *three-term Gale-Robinson recurrence.*
The *Somos-4 sequence*, s_n, a special case of a family of sequences introduced by Michael Somos, is a three-term Gale-Robinson sequence with the following conditions:

$$m = 4, \qquad i = 1, \qquad j = 3, \qquad k = l = 2, \qquad s_0 = s_1 = s_2 = s_3 = 1.$$

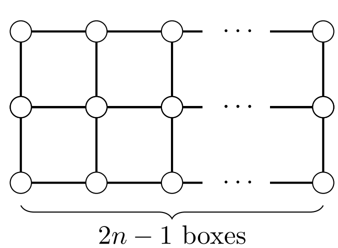
7. **[1]** Calculate s_4, s_5, s_6, and s_7.

8. **[10]** Prove that all terms of the Somos-4 sequence are integers.

9. **[10]** Suppose instead that we still have $m = 4$, $i = 1$, $j = 3$, $k = l = 2$, but different initial values a_0, a_1, a_2, a_3. However, $a_0, ..., a_7$ are integers. Let a_i be written in the form $\frac{n_i}{d_i}$, where n_i, d_i are integers and $\gcd(n_i, d_i) = 1$. Show that for any natural number i and any prime $p|d_i$, we have $p| \gcd(a_2, a_3, a_4)$.

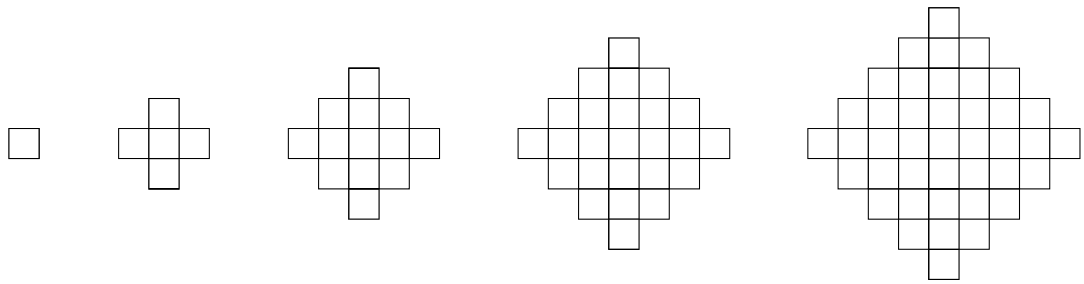
4. Tying It All Together

10. **[8]** Show that the number of matchings g_n for the following graph



satisfies the recurrence $g_n g_{n-2} = g_{n-1}^2 + 2$.

We consider the *Aztec Diamond graphs*, which consist of a row of 1 square centered atop a row of 3 squares, ..., a row of $2n-1$ squares, then symmetrically at the bottom. The first few Aztec Diamonds are shown here:



11. **[4]** Show that every perfect matching of an Aztec Diamond must contain either both the topmost and bottommost edges, or both the leftmost and rightmost edges.

12. (a) **[12]** The number of perfect matchings on the Aztec Diamond of size n satisfies a Gale-Robinson recurrence. Find the initial conditions, and the i, j, k and l for this sequence.

    (b) **[8]** Find an explicit formula for the number of perfect matchings of an Aztec Diamond of size n.