TEAM ROUND

1. Let S be the set of all rational numbers $x \in [0,1]$ with repeating base 6 expansion $x = 0.\overline{a_1 a_2 \cdots a_k} = 0.a_1 a_2 \cdots a_k a_1 a_2 \cdots a_k \cdots$ for some finite sequence $\{a_i\}_{i=1}^k$ of distinct nonnegative integers less than 6. What is the sum of all numbers that can be written in this form? (Put your answer in base 10.)

2. Evaluate
$$\prod_{k=1}^{254} \log_{k+1}(k+2)^{u_k}, \qquad \text{where } u_k = \begin{cases} -k & \text{if } k \text{ is odd} \\ \frac{1}{k-1} & \text{if } k \text{ is even} \end{cases}$$

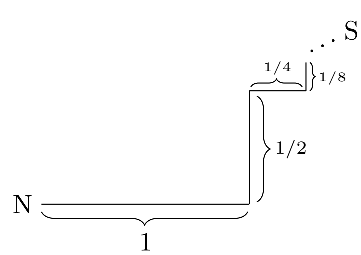
3. Let ABC be a triangle with side lengths $AB = 2011$, $BC = 2012$, $AC = 2013$. Create squares $S_1 = ABB'A''$, $S_2 = ACC''A'$, and $S_3 = CBB''C'$ using the sides AB, AC, BC respectively, so that the side $B'A''$ is on the opposite side of AB from C, and so forth. Let square S_4 have side length $A''A'$, square S_5 have side length $C''C'$, and square S_6 have side length $B''B'$. Let $A(S_i)$ be the area of square S_i. Compute $\frac{A(S_4)+A(S_5)+A(S_6)}{A(S_1)+A(S_2)+A(S_3)}$?

4. There are 12 people labeled $1, \ldots, 12$ working together on 12 missions, with people $1, \ldots, i$ working on the ith mission. There is exactly one spy among them. If the spy is not working on a mission, it will be a huge success, but if the spy is working on the mission, it will fail with probability 1/2. Given that the first 11 missions succeed, and the 12th mission fails, what is the probability that person 12 is the spy?

5. Let $p > 1$ be relatively prime to 10. Let n be any positive number and d be the last digit of n. Define $f(n) = \lfloor \frac{n}{10} \rfloor + d \cdot m$. Then, we can call m a divisibility multiplier for p, if $f(n)$ is divisible by p if and only if n is divisible by p. Find a divisibility multiplier for 2013.

6. A circle with diameter $\overline{AB}$ is drawn, and the point P is chosen on segment $\overline{AB}$ so that $\frac{AP}{AB} = \frac{1}{42}$. Two new circles a and b are drawn with diameters $\overline{AP}$ and $\overline{PB}$ respectively. The perpendicular line to $\overline{AB}$ passing through P intersects the circle twice at points S and T. Two more circles s and t are drawn with diameters $\overline{SP}$ and $\overline{ST}$ respectively. For any circle ω let $A(\omega)$ denote the area of the circle. What is $\frac{A(s)+A(t)}{A(a)+A(b)}$?

7. Suppose Bob begins walking at a constant speed from point N to point S along the path indicated by the following figure.



After Bob has walked a distance of x, Alice begins walking at point N, heading towards point S along the same path. Alice walks 1.28 times as fast as Bob when they are on the same line segment and 1.06 times as fast as Bob otherwise. For what value of x do Alice and Bob meet at point S?

8. Let φ be the Euler totient function. Let $\varphi^k(n) = \underbrace{(\varphi \circ \cdots \circ \varphi)}_{k}(n)$ be φ composed with itself k times. Define $\theta(n) = \min\left\{k \in \mathbb{N} \mid \varphi^k(n) = 1\right\}$. For example,
$$\begin{aligned}\varphi^1(13) &= \varphi(13) = 12\\ \varphi^2(13) &= \varphi(\varphi(13)) = 4\\ \varphi^3(13) &= \varphi(\varphi(\varphi(13))) = 2\\ \varphi^4(13) &= \varphi(\varphi(\varphi(\varphi(13)))) = 1\end{aligned}$$
so $\theta(13) = 4$. Let $f(r) = \theta(13^r)$. Determine $f(2012)$.

9. A permutation of a set is a bijection from the set to itself. For example, if σ is the permutation $1 \mapsto 3$, $2 \mapsto 1$, and $3 \mapsto 2$, and we apply it to the ordered triplet $(1, 2, 3)$, we get the reordered triplet $(3, 1, 2)$. Let σ be a permutation of the set $\{1, \ldots, n\}$. Let
$$\theta_k(m) = \begin{cases} m+1 & \text{for } m < k \\ 1 & \text{for } m = k \\ m & \text{for } m > k \end{cases}$$
Call a finite sequence $\{a_i\}_{i=1}^{j}$ a disentanglement of σ if $\theta_{a_j} \circ \cdots \circ \theta_{a_1} \circ \sigma$ is the identity permutation. For example, when $\sigma = (3, 2, 1)$, then $\{2, 3\}$ is a disentaglement of σ. Let $f(\sigma)$ denote the minimum number k such that there is a disentanglement of σ of length k. Let $g(n)$ be the expected value for $f(\sigma)$ if σ is a random permutation of $\{1, \ldots, n\}$. What is $g(6)$?

10. Suppose that 728 coins are set on a table, all facing heads up at first. For each iteration, we randomly choose 314 coins and flip them (from heads to tails or vice versa). Let $\frac{a}{b}$ be the expected number of heads after we finish 4001 iterations, where a and b are relatively prime. Find $a + b$ mod 10000.