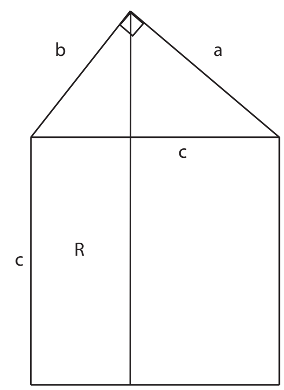
Round 1

1. Find all prime factors of 8051.
2. Simplify
$$[\log_{xyz}(x^z)][1+\log_x y+\log_x z],$$
where $x=628, y=233, z=340$.
3. In prokaryotes, translation of mRNA messages into proteins is most often initiated at start codons on the mRNA having the sequence AUG. Assume that the mRNA is single-stranded and consists of a sequence of bases, each described by a single letter A,C,U, or G.

   Consider the set of all pieces of bacterial mRNA six bases in length. How many such mRNA sequences have either no A's or no U's?
4. What is the smallest positive n so that 17^n+n is divisible by 29?
5. The legs of the right triangle shown below have length $a=255$ and $b=32$. Find the area of the smaller rectangle (the one labeled R).



6. A 3 dimensional cube contains "cubes" of smaller dimensions, ie: faces (2-cubes),edges (1-cubes), and vertices (0-cubes). How many 3-cubes are in a 5-cube?

Round 2

1. 4 balls are distributed uniformly at random among 6 bins. What is the expected number of empty bins?
2. Compute $\binom{150}{20} \pmod{221}$.
3. On the right triangle ABC, with right angle at B, the altitude BD is drawn. E is drawn on BC such that AE bisects angle BAC and F is drawn on AC such that BF bisects angle CBD. Let the intersection of AE and BF be G. Given that $AB = 15, BC = 20, AC = 25$, find $\frac{BG}{GF}$.
4. What is the largest integer n so that $\frac{n^2-2012}{n+7}$ is also an integer?
5. What is the side length of the largest equilateral triangle that can be inscribed in a regular pentagon with side length 1?
6. Inside a LilacBall, you can find one of 7 different notes, each equally likely. Delcatty must collect all 7 notes in order to restore harmony and save Kanto from eternal darkness. What is the expected number of LilacBalls she must open in order to do so?

Round 3

1. Let $A(S)$ denote the average value of a set S. Let T be the set of all subsets of the set $\{1, 2, 3, 4, \ldots, 2012\}$, and let R be $\{A(K)|K \in T\}$. Compute $A(R)$

2. Consider the minute and hour hands of the Campanile, our clock tower. During one single day (12:00 AM - 12:00 AM), how many times will the minute and hour hands form a right-angle at the center of the clock face?

3. In a regular deck of 52 face-down cards, Billy flips 18 face-up and shuffles them back into the deck. Before giving the deck to Penny, Billy tells her how many cards he has flipped over, and blindfolds her so she can't see the deck and determine where the face-up cards are. Once Penny is given the deck, it is her job to split the deck into two piles so that both piles contain the same number of face-up cards. Assuming that she knows how to do this, how many cards should be in each pile when he is done?

4. The roots of the equation $x^3 + ax^2 + bx + c = 0$ are three consecutive integers. Find the maximum value of $\frac{a^2}{b+1}$

5. Oski has a bag initially filled with one blue ball and one gold ball. He plays the following game: first, he removes a ball from the bag. If the ball is blue, he will put another blue ball in the bag with probability $\frac{1}{437}$ and a gold ball in the bag the rest of the time. If the ball is gold, he will put another gold ball in the bag with probability $\frac{1}{437}$ and a blue ball in the bag the rest of the time. In both cases, he will put the ball he drew back into the bag. Calculate the expected number of blue balls after 525600 iterations of this game.

6. Circles A and B intersect at points C and D. Line AC and circle B meet at E, line BD and circle A meet at F, and lines EF and AB meet at G. If $AB = 10, EF = 4, FG = 8$, find BG.

Round 4

1. Denote $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$. What is $144169 \cdot S_{144169} - (S_1 + S_2 + \ldots + S_{144168})$?

2. Let A, B, C be three collinear points, with $AB = 4$, $BC = 8$, and $AC = 12$. Draw circles with diameters AB, BC, and AC. Find the radius of the two identical circles that will lie tangent to all three circles.

3. Let $s(i)$ denote the number of 1's in the binary representation of i. What is $\sum_{x=1}^{314} \left(\sum_{i=0}^{2^{576}-2} x^{s(i)} \right) \bmod 629$?

4. Parallelogram $ABCD$ has an area of S. Let $k = 42$. E is drawn on AB such that $AE = \frac{AB}{k}$. F is drawn on CD such that $CF = \frac{CD}{k}$. G is drawn on BC such that $BG = \frac{BC}{k}$. H is drawn on AD such that $DH = \frac{AD}{k}$. Line CE intersects BH at M, and DG at N. Line AF intersects DG at P, and BH at Q. If S_1 is the area of quadrilateral $MNPQ$, find $\frac{S_1}{S}$

5. Let φ be the Euler totient function. What is the sum of all n for which $\frac{n}{\varphi(n)}$ is maximal for $1 \leq n \leq 500$?

6. Link starts at the top left corner of an 12×12 grid and wants to reach the bottom right corner. He can only move down or right. A turn is defined a down move immediately followed by a right move, or a right move immediately followed by a down move. Given that he makes exactly 6 turns, in how many ways can he reach his destination?

Round 4

1. Let n be the number so that $1 - 2 + 3 - 4 + \ldots - (n-1) + n = 2012$. What is $4^{2012} \pmod{n}$?

2. Consider three unit squares placed side by side. Label the top left vertex P and the bottom four vertices A, B, C, D respectively. Find $\angle PBA + \angle PCA + \angle PDA$.

3. Given $f(x) = \frac{3}{x-1}$, then express $\frac{9(x^2-2x+1)}{x^2-8x+16}$ entirely in terms of $f(x)$. In other words, x should not be in your answer, only $f(x)$.

4. Right triangle with right angle B and integer side lengths has BD as the altitude. E and F are the incenters of triangles ADB and BDC respectively. Line EF is extended and intersects BC at G, and AB at H. If $AB = 15$ and $BC = 8$, find the area of triangle BGH.

5. Let $a_1, a_2, \ldots, a_n$ be a sequence of real numbers. Call a k-inversion $(0 < k \leq n)$ of a sequence to be indices $i_1, i_2, \ldots, i_k$ such that $i_1 < i_2 < \ldots < i_k$ but $a_{i_1} > a_{i_2} > \ldots > a_{i_k}$. Calculate the expected number of 6-inversions in a random permutation of the set $\{1, 2, \ldots, 10\}$.

6. Chell is given a strip of squares labeled $1, \ldots, 6$ all placed side to side. For each $k \in \{1, ..., 6\}$, she then chooses one square at random in $\{1, ..., k\}$ and places a Weighted Storage Cube there. After she has placed all 6 cubes, she computes her score as follows: For each square, she takes the number of cubes in the pile and then takes the square (i.e. if there were 3 cubes in a square, her score for that square would be 9). Her overall score is the sum of the scores of each square. What is the expected value of her score?

Championship Round

1. If n is a positive integer such that $2n+1 = 144169^2$, find two consecutive numbers whose squares add up to $n+1$.

2. Katniss has an n sided fair die which she rolls. If $n > 2$, she can either choose to let the value rolled be her score, or she can choose to roll a $n-1$ sided fair die, continuing the process. What is the expected value of her score assuming Katniss starts with a 6 sided die and plays to maximize this expected value?

3. Suppose that $f(x) = x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f$, and that $f(1) = f(2) = f(3) = f(4) = f(5) = f(6) = 7$. What is a?

4. a and b are positive integers so that $20a + 12b$ and $20b - 12a$ are both powers of 2, but $a + b$ is not. Find the minimum possible value of $a + b$.

5. Square $ABCD$ and rhombus $CDEF$ share a side. If $m\angle DCF = 36^\circ$, find the measure of $\angle AEC$.

6. Tom challenges Harry to a game. Tom first blindfolds Harry and begins to set up the game. Tom places 4 quarters on an index card, one on each corner of the card. It is Harry's job to flip all the coins either face-up or face-down using the following rules:

   (a) Harry is allowed to flip as many coins as he wants during his turn.

   (b) A turn consists of Harry flipping as many coins as he wants (while blindfolded). When he is happy with what he has flipped, Harry will ask Tom whether or not he was successful in flipping all the coins face-up or face-down. If yes, then Harry wins. If no, then Tom will take the index card back, rotate the card however he wants, and return it back to Harry, thus starting Harry's next turn. Note that Tom cannot touch the coins after he initially places them before starting the game.

   Assuming that Tom's initial configuration of the coins weren't all face-up or face-down, and assuming that Harry uses the most efficient algorithm, how many moves maximum will Harry need in order to win? Or will he never win?

Consolation Round

1. How many ways can we arrange the elements $\{1, 2, \ldots, n\}$ to a sequence $a_1, a_2, \ldots, a_n$ such that there is only exactly one a_i, a_{i+1} such that $a_i > a_{i+1}$?

2. How many distinct (non-congruent) triangles are there with integer side-lengths and perimeter 2012?

3. Let φ be the Euler totient function, and let $S = \{x | \frac{x}{\varphi(x)} = 3\}$. What is $\sum_{x \in S} \frac{1}{x}$?

4. Denote $f(N)$ as the largest odd divisor of N. Compute $f(1) + f(2) + f(3) + \ldots + f(29) + f(30)$.

5. Triangle ABC has base AC equal to 218 and altitude 100. Squares $s_1, s_2, s_3, \ldots$ are drawn such that s_1 has a side on AC and has one point each touching AB and BC, and square s_k has a side on square s_{k-1} and also touches AB and BC exactly once each. What is the sum of the area of these squares?

6. Let P be a parabola $6x^2 - 28x + 10$, and F be the focus. A line l passes through F and intersects the parabola twice at points $P_1 = (2, -22)$, P_2. Tangents to the parabola with points at P_1, P_2 are then drawn, and intersect at a point Q. What is $m\angle P_1 Q P_2$?