1. Let

$$f(x) = \frac{x^{2020}}{2020} + 2020!.$$

Compute $f''(1)$.

Answer: 2019

Solution: Since $f'(x) = x^{2019}$, $f''(x) = 2019x^{2018}$; therefore, $f''(1) = \boxed{2019}$.

2. Compute the integral

$$\int_{-20}^{20} (20 - |x|)\,\mathrm{d}x\,.$$

Answer: 400

Solution: We have

$$\begin{aligned}\int_{-20}^{20} (20 - |x|)\,\mathrm{d}x &= 2\int_{0}^{20} (20 - x)\,\mathrm{d}x \\ &= 2\int_{0}^{20} x\,\mathrm{d}x \\ &= \left[x^2\right]_0^{20} \\ &= \boxed{400}.\end{aligned}$$

3. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a differentiable function defined by

$$f(x) = f'(2)x^2 + x.$$

Then $f(2)$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

Answer: 5

Solution: Taking the derivative on both sides gives us

$$\begin{aligned}f'(x) &= 2f'(2)x + 1 \\ f'(2) &= 2f'(2) \cdot 2 + 1 \\ 3f'(2) &= -1 \\ f'(2) &= -\frac{1}{3} \\ f(2) &= 4f'(2) + 2 \\ &= -\frac{4}{3} + 2 \\ &= \frac{2}{3},\end{aligned}$$

and our answer is $\boxed{5}$.

4. If a is a positive real number such that the region of finite area bounded by the curve $y = x^2 + 2020$, the line tangent to that curve at $x = a$, and the y-axis has area 2020, compute a^3.

Answer: 6060

Solution: Taking the derivative of $f(x) = x^2 + 2020$, we have $f'(x) = 2x$, so $f'(a) = 2a$. Then the tangent line at $x = a$ to $f(x)$ has slope $2a$. Let the tangent line be $g(x) = 2ax + c$, then we must have $f(a) = g(a)$, giving $2a^2 + c = a^2 + 2020$, so $c = 2020 - a^2$. Then the tangent line has equation $y = 2ax - a^2 + 2020$, so the area of that region is

$$\begin{aligned} 2020 &= \int_0^a \left(x^2 - 2ax + a^2\right) \mathrm{d}x \\ &= \int_0^a (x-a)^2 \, \mathrm{d}x \\ &= \frac{a^3}{3} \\ a^3 &= \boxed{6060}. \end{aligned}$$

5. Suppose that a parallelogram has a vertex at the origin of the 2-dimensional plane, and two of its sides are vectors from the origin to the points $(10, y)$, and $(x, 10)$, where $x, y \in [0, 10]$ are chosen uniformly at random. The probability that the parallelogram's area is at least 50 is $\ln(\sqrt{a}) + \frac{b}{c}$, where a, b, and c are positive integers such that b and c are relatively prime and a is as small as possible. Compute $a + b + c$.

Answer: 5

Solution: The parallelogram's area is given by the determinant of the matrix $\begin{bmatrix} 10 & y \\ x & 10 \end{bmatrix}$ which is equal to $100 - xy$. It thus suffices to find the probability that $xy \leq 50$ given $x, y \in [0, 10]$. This simplifies to $y \leq \dfrac{50}{x}$. Then if $xy \leq 50$ then either $x < 5$, with probability $\frac{1}{2}$ by symmetry, or $xy > 50$ given $x \geq 5$, and this probability is $\frac{1}{100} \int_5^{10} \frac{50}{x} \, \mathrm{d}x$ (we divide by 100 to normalize the probability, since the integral computes the area). This integral evaluates to $\frac{1}{100}(50 \ln(2)) = \frac{\ln(2)}{2}$, so the total probability is $\frac{1}{2} + \frac{\ln(2)}{2} = \frac{1+\ln(2)}{2}$, and therefore our answer is $\boxed{5}$.

6. For some $a > 1$, the curves $y = a^x$ and $y = \log_a(x)$ are tangent to each other at exactly one point. Compute $|\ln(\ln(a))|$.

Answer: 1

Solution: Since the functions a^x and $\log_a(x)$ are inverses, by symmetry their point of tangency occurs on the line $y = x$. Let the intersection point be (z, z); at this point, again by symmetry, $a^z = \log_a(z)$, and

$$\frac{\mathrm{d}}{\mathrm{d}z} a^z = a^z \ln(a) = \frac{\mathrm{d}}{\mathrm{d}z} \log_a(z) = \frac{1}{z \ln(a)} = 1.$$

Substituting $a^z = \log_a(z)$ into $a^z \ln(a) = 1$, we obtain $\ln(a) \log_a(z) = 1$, so $\ln(z) = 1$, so $z = \mathrm{e}$. Substituting $z = \mathrm{e}$ back into the equation $\frac{1}{z \ln(a)} = 1$, we have $\frac{1}{\mathrm{e} \ln(a)} = 1$, so $a = \mathrm{e}^{1/\mathrm{e}}$ and $|\ln(\ln(a))| = \boxed{1}$ as desired.

7. The limit

$$\lim_{n \to \infty} n^2 \int_0^{1/n} x^{x+1} \, \mathrm{d}x$$

can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

Answer: 3

Solution: Let $h = \frac{1}{n}$, so

$$\begin{aligned}
\lim_{n\to\infty} n^2 \int_0^{1/n} x^{x+1}\,\mathrm{d}x &= \lim_{h\to 0^+} \frac{\int_0^h x^{x+1}\,\mathrm{d}x}{h^2} \\
&= \lim_{h\to 0^+} \frac{\frac{\mathrm{d}}{\mathrm{d}h}\int_0^h x^{x+1}\,\mathrm{d}x}{\frac{\mathrm{d}}{\mathrm{d}h}h^2} && (\dagger) \\
&= \lim_{h\to 0^+} \frac{h^{h+1} - 0^1}{2h} \\
&= \frac{1}{2}\lim_{h\to 0^+} h^h \\
&= \frac{1}{2}\exp\left(\lim_{h\to 0^+} \ln\left(h^h\right)\right) \\
&= \frac{1}{2}\exp\left(\lim_{h\to 0^+} h\ln(h)\right) \\
&= \frac{1}{2}\exp\left(\lim_{h\to 0^+} \frac{\ln(h)}{1/h}\right) \\
&= \frac{1}{2}\exp\left(\lim_{h\to 0^+} \frac{\frac{\mathrm{d}}{\mathrm{d}h}\ln(h)}{\frac{\mathrm{d}}{\mathrm{d}h}\frac{1}{h}}\right) && (\dagger) \\
&= \frac{1}{2}\exp\left(\lim_{h\to 0^+} \frac{1/h}{-1/h^2}\right) \\
&= \frac{1}{2}\exp\left(\lim_{h\to 0^+} -h\right) \\
&= \frac{1}{2}\exp(0) \\
&= \frac{1}{2},
\end{aligned}$$

and our answer is $\boxed{3}$. Here, the statements marked with (†) are derived using L'Hopital's rule, and the rest is algebraic manipulation.

8. The summation

$$\sum_{a=1}^{\infty}\sum_{b=1}^{\infty} \frac{1}{a^2b + 2ab + ab^2}$$

can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.

Answer: 11

Solution: We have

$$\begin{aligned}
\frac{1}{a^2b+2ab+ab^2} &= \frac{1}{ab(a+b+2)} \\
&= \frac{1}{ab}\int_0^1 x^{a+b+1}\,\mathrm{d}x \\
\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\frac{1}{a^2b+2ab+ab^2} &= \sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\frac{1}{ab}\int_0^1 x^{a+b+1}\,\mathrm{d}x \\
&= \int_0^1 x\left(\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\frac{x^{a+b}}{ab}\right)\mathrm{d}x \qquad \text{(Interchange integral and summation.)} \\
&= \int_0^1 x\left(\sum_{a=1}^{\infty}\frac{x^a}{a}\right)\left(\sum_{b=1}^{\infty}\frac{x^b}{b}\right)\mathrm{d}x \\
&= \int_0^1 x\left(\sum_{a=1}^{\infty}\frac{x^a}{a}\right)^2\mathrm{d}x \\
&= \int_0^1 x\ln(1-x)^2\,\mathrm{d}x \qquad \text{(Taylor series of } \ln(1-x).\text{)} \\
&= \int_0^1 (1-x)\ln(x)^2\,\mathrm{d}x \qquad \text{(Use the transformation } x\to 1-x.\text{)} \\
&= \frac{7}{4}. \qquad \text{(Using integration by parts.)}
\end{aligned}$$

And so our answer is $\boxed{11}$.

9. Let $f\colon \mathbb{R}_{>0}\to\mathbb{R}$ (where $\mathbb{R}_{>0}$ is the set of all positive real numbers) be differentiable and satisfy the equation

$$f(y)-f(x)=\frac{x^x}{y^y}f\left(\frac{y^y}{x^x}\right)$$

for all real $x, y > 0$. Furthermore, $f'(1)=1$. Compute $\frac{f(2020^2)}{f(2020)}$.

Answer: 4040

Solution: Set $x = y = 1$; we obtain $f(1) = 0$. Applying the definition of a derivative,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{x^x}{(x+h)^{x+h}} f\left(\frac{(x+h)^{x+h}}{x^x}\right)}{h} \\
&= \lim_{h\to 0} \frac{f\left(\frac{(x+h)^{x+h}}{x^x}\right)}{h} && \left(\text{Since } \lim_{h\to 0} \tfrac{x^x}{(x+h)^{x+h}} = 1.\right) \\
&= \lim_{h\to 0} \frac{f\left(\frac{(x+h)^{x+h}}{x^x}\right) - f(1)}{h} && (\text{Since } f(1) = 0.) \\
f'(1) &= \lim_{h\to 0} \frac{f\left(\frac{(x+h)^{x+h}}{x^x}\right) - f(1)}{\frac{(x+h)^{x+h}}{x^x} - 1} \\
&= 1 \\
\lim_{h\to 0} \left(\frac{(x+h)^{x+h}}{x^x} - 1\right) &= \lim_{h\to 0} \left(f\left(\frac{(x+h)^{x+h}}{x^x}\right) - f(1)\right) \\
f'(x) &= \lim_{h\to 0} \frac{\frac{(x+h)^{x+h}}{x^x} - 1}{h}
\end{aligned}
$$

This limit is nontrivial, but possible to evaluate.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\frac{(x+h)^{x+h}}{x^x} - 1}{h} \\
&= \lim_{h\to 0} \frac{x^{-x}(x+h)^{x+h} - 1}{h} \\
&= \lim_{h\to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}h}\left[x^{-x}(x+h)^{x+h} - 1\right]}{\frac{\mathrm{d}}{\mathrm{d}h}h} \\
&= \lim_{h\to 0} \frac{x^{-x}(x+h)^{x+h}(\ln(x+h) + 1)}{1} \\
&= \lim_{h\to 0} x^{-x}(x+h)^{x+h}(\ln(x+h) + 1) \\
&= x^{-x}(x+0)^{x+0}(\ln(x+0) + 1) \\
&= \ln(x) + 1
\end{aligned}
$$

Now that we obtain $f'(x) = \ln(x) + 1$, we integrate to find $f(x)$:

$$
\begin{aligned}
\int (\ln(x) + 1)\,\mathrm{d}x &= \int \ln(x)\,\mathrm{d}x + \int \mathrm{d}x \\
&= x\ln(x) - \int \mathrm{d}x + \int \mathrm{d}x \\
&= x\ln(x) + C
\end{aligned}
$$

where C is some constant and $\int \ln(x)\,\mathrm{d}x$ is evaluated by integration parts, where $u = \log(x)$ (so $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$) and $\mathrm{d}v = \mathrm{d}x$ (so $v = x$).

Using the initial condition $f(1) = [x\ln(x) + C]_{x=1} = 1\ln(1) + C = C = 0$, we obtain $f(x) = x\ln(x)$ and $\frac{f(2020^2)}{f(2020)} = \frac{2020^2 \cdot \ln(2020^2)}{2020 \cdot \ln(2020)} = \boxed{4040}$.

10. The integral

$$\int_0^{\frac{\pi}{2}} \frac{x}{\tan(x)}\,\mathrm{d}x$$

can be written in the form $a^b \pi \ln c$, where a, b, and c are integers such that c is as small as possible. Compute $a + b + c$.

Answer: 3

Solution: Using the Feynman trick (differentiate under the integral sign), we rewrite the integral as

$$\begin{aligned}
I(a) &= \int_0^{\pi/2} \frac{\arctan(a\tan(x))}{\tan(x)}\,\mathrm{d}x \\
I'(a) &= \int_0^{\pi/2} \frac{1}{1 + a^2\tan^2(x)}\,\mathrm{d}x \\
&= \int_0^{\pi/2} \frac{\sec^2(x)}{(1 + a^2\tan^2(x))\,(1 + \tan^2(x))}\,\mathrm{d}x \\
&= \frac{1}{a^2 - 1}\int_0^{\pi/2} \left(\frac{a^2\sec^2(x)}{1 + a^2\tan^2(x)} - \frac{\sec^2(x)}{1 + \tan^2(x)}\right)\mathrm{d}x \\
&= \frac{1}{a^2 - 1}[a\arctan(a\tan(x)) - x]_0^{\pi/2} \\
&= \frac{\pi}{2} \cdot \frac{1}{a + 1} \\
I(a) &= \frac{\pi}{2}\ln(a + 1) + C \\
I(0) &= \int_0^{\pi/2} \frac{\arctan(0)}{\tan(x)}\,\mathrm{d}x \\
&= 0 \\
&= C \\
\int_0^{\pi/2} \frac{x}{\tan(x)}\,\mathrm{d}x &= I(1) \\
&= \frac{\pi}{2}\ln(2) + C \\
&= \frac{\pi}{2}\ln(2),
\end{aligned}$$

which is equal to $2^{-1}\pi \ln 2$ (and is the only way to represent our answer in the given form). Our answer, therefore, is $2 + (-1) + 2 = \boxed{3}$.