1. Let the sequence $\{a_n\}$ for $n \geq 0$ be defined as $a_0 = c$, and for $n \geq 0$,

$$a_n = \frac{2a_{n-1}}{4a_{n-1}^2 - 1}.$$

Compute the sum of all values of c such that a_{2020} exists but a_{2021} does not exist.

Answer: 0

Solution 1: Note that if $a_0 = c$ works, then $a_0 = -c$ also works, as the sequence given by $a_0 = -c$ is just the sequence given by $a_0 = c$ with all terms in the sequence negated. Thus, our answer is $\boxed{0}$.

Solution 2: Define the sequence $\{\theta_n\}$ such that $a_n = \frac{1}{2}\tan\theta_n$. Then we have that $\theta_n = 2^n\theta_0$.

For a_{2021} not to exist while a_{2020} exists, we need that $\theta_{2021} = \frac{\pi}{2} + \pi m$ for some integer m. Thus, $\theta_0 = \frac{(2m+1)}{2^{2022}}$ for some integer m.

Summing over all m such that θ_0 is in the range $[0, 2\pi)$, we get the sum of $\boxed{0}$, since symmetric terms cancel out.

2. Real numbers x and y satisfy the equations $x^2 - 12y = 17^2$ and $38x - y^2 = 2 \cdot 7^3$. Compute $x + y$.

Answer: 25

Solution: Subtracting the second equation from the first equation gives $x^2 - 38x + y^2 - 12y = -397$, and completing the square for x and y gives $(x - 19)^2 + (y - 6)^2 = 0$. Since $a^2 = 0$ if and only if $a = 0$, we must have $x = 19$ and $y = 6$, so $x + y = \boxed{25}$. (We can check that this solution works by plugging back into the two original equations.)

3. For integers a and b, $a + b$ is a root of $x^2 + ax + b = 0$. Compute the smallest possible value of ab.

Answer: −54

Solution: By Vieta's formulas, the two roots are $x = -2a - b$ and $x = a + b$. Moreover, we can build and simplify an equation regarding their product: $(a + b)(-2a - b) = b$, so

$$2a^2 + 3ba + (b^2 + b) = 0.$$

We further establish a condition based on the fact that the roots are integers that the discriminant must be equivalent to the square of an integer k:

$$\Delta_1 = (3b)^2 - 4 \cdot 2(b^2 + b) = k^2.$$

It follows that $b^2 - 8b - k^2 = 0$. We can take the discriminant of this new quadratic with respect to b again:

$$\Delta_2 = 8^2 + 4k^2 = 64 + 4k^2 = q^2,$$

for some integer q. Let $q = 2m$ so that $64 + 4k^2 = 4m^2$; it follows that $16 = (m + k)(m - k)$.

We generate ordered pairs (m, k) that satisfy this constraint by difference of squares; we get the pairs $(4, 0)$, $(-4, 0)$, $(5, -3)$, $(5, 3)$, $(-5, -3)$, and $(-5, 3)$. We have values of $k = 0, \pm 3$, which we can substitute into our previous equations for b to get $b^2 - 8b = 0$ (which gives $b = 0, 8$) and $b^2 - 8b - 9 = 0$ (which gives $b = 9, -1$).

With these values for b, we can substitute back into $2a^2+3ba+(b^2+b)=0$ to get the quadratics

$$\begin{cases} 2a^2 = 0 & b = 0, \\ 2a^2 - 3a = 0 & b = -1, \\ 2a^2 + 24a + 72 = 0 & b = 8, \\ 2a^2 + 27a + 90 = 0 & b = 9. \end{cases}$$

Solving each of these for a shows that $b=9$ yields the smallest value of $ab = (-6)(9) = \boxed{-54}$.