1. Let $g(x) = \int_{2021}^{x}(e^t - 2t)\,\mathrm{d}t$. Compute $g'(2021)$.

   Answer: $\boldsymbol{e^{2021} - 4042}$

   Solution: By the Fundamental Theorem of Calculus, $g'(x) = e^x - 2x$. So, the answer is $\boxed{e^{2021} - 4042}$.

2. Let $f(x) = (x+3)(2x+5)(3x+7)(x+1)$. Compute $f^{(4)}(5)$. (Note that $f^{(4)}(5) = f''''(5)$.)

   Answer: **144**

   Solution: Note that $f(x)$ is a quartic, so taking the fourth derivative gives us a constant, and we only need to care about the x^4 term. This term ends up being $6x^4$, and its fourth derivative is $6 \cdot 4! = \boxed{144}$, our answer.

3. A quadratic function in the form $x^2 + cx + d$ has vertex (a, b). If this function and its derivative are graphed on the coordinate plane, then they intersect at exactly one point. Compute b.

   Answer: **1**

   Solution: The quadratic function with vertex (a, b) is $(x-a)^2 + b$, and its derivative is $2(x-a)$. When we set them equal, we expect the resulting equation $(x-a)^2 + b = 2(x-a)$ to have exactly one solution. Moving all terms to the left side, $(x-a)^2 - 2(x-a) + b = 0$, and now we can complete the square: $((x-a)-1)^2 - 1 + b = 0$. So $(x-a-1)^2 = -b+1$, and for this equation to have exactly one solution, we must have $-b+1 = 0$, or $b = \boxed{1}$.

4. Compute the area of the region of points satisfying the inequalities $y \le 4 - \frac{x^2}{9}$, $y \ge \frac{x^2}{9} - 4$, $x \le 4 - \frac{y^2}{9}$, and $x \ge \frac{y^2}{9} - 4$.

   Answer: **52**

   Solution: The region enclosed by these parabolas is a square with extra parabola lumps of equal size, with vertices at the intersections of the parabolas at $(\pm 3, \pm 3)$. The area of a parabola lump is $\int_{-3}^{3}((4 - \frac{x^2}{9}) - 3)\,\mathrm{d}x = 4$, and the area of the square is $6^2 = 36$, so the area of the region is $36 + 4 \cdot 4 = \boxed{52}$.

5. Suppose the following equality holds, where a, b, c are integers and K is the constant of integration:

$$\int \frac{\sin^a(x) - \cos^a(x)}{\sin^b(x)\cos^b(x)}\,\mathrm{d}x = \frac{\csc^c(x)}{c} + \frac{\sec^c(x)}{c} + K.$$

   If $a = 2021$, compute $a + b + c$.

   Answer: **6060**

   Solution: We see that

$$\int \frac{\sin^a(x) - \cos^a(x)}{\sin^b(x)\cos^b(x)}\,\mathrm{d}x = \int \left(\frac{\sin^{a-b} x}{\cos^b x} - \frac{\cos^{a-b} x}{\sin^b x}\right)\mathrm{d}x\,,$$

   and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\csc^c x}{c} + \frac{\sec^c x}{c}\right] = -\csc^c x \cot x + \sec^c x \tan x = -\frac{\cos x}{\sin^{c+1} x} + \frac{\sin x}{\cos^{c+1} x},$$

   so

$$-\frac{\cos x}{\sin^{c+1} x} + \frac{\sin x}{\cos^{c+1} x} = \frac{\sin^{a-b} x}{\cos^b x} - \frac{\cos^{a-b} x}{\sin^b x}.$$

This gives us a system of equations

$$\begin{cases} a-b=1, \\ c+1=b. \end{cases}$$

By plugging in $a = 2021$, we get $b = 2020$ and $c = 2019$, so $a+b+c = \boxed{6060}$.

6. Let $x_1 = -4$, and for $n \geq 1$, define $x_{n+1} = -4^{x_n}$. Similarly, let $f_1(x) = \sin(\arccos x)$, and for $n \geq 1$, define $f_{n+1}(x) = f_1(f_n(x))$. Compute

$$\lim_{n\to\infty} f_n(2^{x_n}).$$

You may assume that this limit exists.

Answer: $\frac{1}{\sqrt{2}}$

Solution: We start by finding $\lim_{n\to\infty} x_n$. Say that this limit evaluates to x. We are given that

$$x_{n+1} = -4^{x_n}.$$

Taking the limit of both sides as $n \to \infty$,

$$x = -4^x.$$

Drawing the graphs for these functions, we observe that they intersect only once. Hence, there is only one solution for x. After some guess and check, we find that $x = -\frac{1}{2}$ works and must be the unique solution.

We now evaluate the desired limit:

$$\lim_{n\to\infty} f_n(2^{x_n}) = \lim_{n\to\infty} f_n(2^x) = \lim_{n\to\infty} f_n\left(\frac{1}{\sqrt{2}}\right).$$

Note that $f_1\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$, so it can be shown through induction that $f_n\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$ for all integers n.

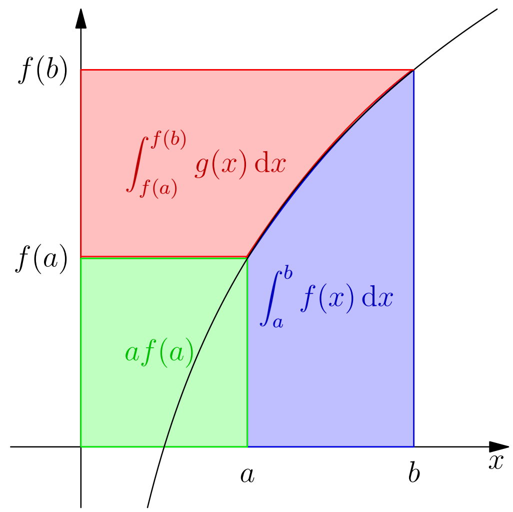
Therefore, we have $\lim_{n\to\infty} f_n(2^{x_n}) = \boxed{\frac{1}{\sqrt{2}}}$.

7. Let $c(x) = \frac{e^x + e^{-2x}}{2}$, defined on the interval $1 \leq x \leq 2$. Let $c^{-1}(x)$ be the inverse of $c(x)$. Compute

$$\int_{c(1)}^{c(2)} c^{-1}(x)\,\mathrm{d}x\,.$$

Answer: $\frac{1}{2}e^2 + \frac{5}{4}e^{-4} - \frac{3}{4}e^{-2}$

Solution: We consider the following identity, which can be derived from a graphical view of the problem below:



Thinking of the integral as an area under the curve, we have:

$$\int_a^b f(x)\,\mathrm{d}x + \int_{f(a)}^{f(b)} g(y)\,\mathrm{d}y + af(a) = bf(b),$$

where g and f are inverse functions.

Thus,

$$\begin{aligned}\int_{c(1)}^{c(2)} c^{-1}(x)\,\mathrm{d}x &= 2c(2) - c(1) - \int_1^2 c(x)\,\mathrm{d}x \\ &= e^2 + e^{-4} - \frac{e+e^{-2}}{2} - \frac{1}{2}\int_1^2 e^x + e^{-2x}\,\mathrm{d}x \\ &= e^2 + e^{-4} - \frac{e}{2} - \frac{e^{-2}}{2} - \frac{e^2}{2} + \frac{e}{2} + \frac{e^{-4}}{4} - \frac{e^{-2}}{4} \\ &= \frac{1}{2}e^2 + \frac{5}{4}e^{-4} - \frac{3}{4}e^{-2}.\end{aligned}$$

Thus our answer is $\boxed{\frac{1}{2}e^2 + \frac{5}{4}e^{-4} - \frac{3}{4}e^{-2}}$.

8. Define

$$f_n(x) = \int_0^x \frac{t^{6n-1}}{1+t^3}\,\mathrm{d}t$$

for positive integers n and real numbers $0 \le x \le 1$. We can write $f_n(x) = c \cdot \log(p(x)) + h_n(x)$, where $p(x)$ and $h_n(x)$ are polynomials with real coefficients with $p(x)$ monic (coefficient of the highest degree term is 1), and c is a real number. Compute

$$\lim_{n\to\infty} h_n(1).$$

Answer: $\mathbf{\frac{\ln 2}{3}}$

Solution: Note that the base of the log does not matter, as value of c changes as the base of the log changes by change of base. We consider the log to be base e.

First, note that $\lim_{n\to\infty} \frac{t^{6n-1}}{t^3+1} = 0$ for $0 \le t < 1$, and $\lim_{n\to\infty} \frac{t^{6n-1}}{t^3+1} = \frac{1}{2}$ for $t = 1$. Taking the integral to be the area under the curve, we have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \int_0^1 \frac{t^{6n-1}}{t^3+1}\,\mathrm{d}t = 0.$$

Therefore,

$$\lim_{n\to\infty} h_n(x) = -\lim_{n\to\infty} c\ln(p(x)).$$

Next, we classify the possible polynomials $p(x)$ could be. Taking the derivative of both sides of $f_n(x) = c \cdot \ln(p(x)) + h_n(x)$ gives

$$\frac{x^{6n-1}}{1+x^3} = f_n'(x) = \frac{cp'(x)}{p(x)} + h_n'(x),$$

or

$$x^{6n-1}p(x) = (1+x^3)(cp'(x) + h_n'(x)p(x)).$$

Then $1+x^3$ must divide $x^{6n-1}p(x)$, but x^{6n-1} and $1+x^3$ do not share any complex roots, so $1+x^3$ must divide $p(x)$. Let $p(x) = (1+x^3)q(x)$, where $q(x)$ is a real (monic) polynomial.

Plugging in, we have that

$$x^{6n-1}(1+x^3)q(x) = (1+x^3)(c((1+x^3)q'(x) + 3x^2q(x)) + h_n'(x)(1+x^3)q(x),$$

so

$$(x^{6n-1} - 3cx^2 - h_n'(x))q(x) = c(1+x^3)q'(x).$$

Suppose $q(x)$ shares no complex roots with $1+x^3$ and has degree ≥ 1. Then $q(x) \mid q'(x)$, which is not possible, as the degree of $q(x)$ is larger than the degree of $q'(x)$. Thus, $q(x)$ must either be 1 or share a root with $1+x^3$. Since $q(x)$ is a real polynomial, $q(x)$ must be of the form $(1+x)^a(1-x+x^2)^b$, where a and b are natural numbers. Equivalently, $p(x) = (1+x)^{a+1}(1-x+x^2)^{b+1}$ for natural numbers a and b.

Plugging this in gives

$$\begin{aligned}
&x^{6n-1}(1+x)^{a+1}(1-x+x^2)^{b+1}\\
=&(1+x^3)(c(1+x)^a(1-x+x^2)^b\left((a+1)(1-x+x^2) + (b+1)(-1+2x)(1+x)\right)\\
&+ h_n'(x)(1+x)^{a+1}(1-x+x^2)^{b+1}),
\end{aligned}$$

and simplifying gives

$$\begin{aligned}
x^{6n-1} &= (c\left((a-b) + (b-a)x + (a+2b+3)x^2\right) + h_n'(x)(1+x^3)\\
\implies h_n'(x) &= \frac{x^{6n-1} - (c\left((a-b) + (b-a)x + (a+2b+3)x^2\right)}{1+x^3}\\
\implies h_n'(x) &= \frac{x^{6n-1}+x^2}{1+x^3} + \frac{-x^2 - (c\left((a-b) + (b-a)x + (a+2b+3)x^2\right)}{1+x^3}.
\end{aligned}$$

Since h_n is a polynomial, h_n' must also be a polynomial. Since $\frac{x^{6n-1}+x^2}{1+x^3}$ is a polynomial, we need $\frac{-x^2-(c\left((a-b)+(b-a)x+(a+2b+3)x^2\right)}{1+x^3}$ to be a polynomial; thus, we need the numerator to be 0. Setting all the coefficients to 0, we get that $a = b$ and $-1 - c(a+2b+3) = 0$, and thus $c = -\frac{1}{3(a+1)}$.

Plugging back into our original expression, we have

$$\begin{aligned}\lim_{n\to\infty} h_n(1) &= -\lim_{n\to\infty} c\ln(p(1))\\ &= -\lim_{n\to\infty} -\frac{1}{3(a+1)}\cdot\ln\Big((1+1)^{a+1}(1-1+1^2)^{b+1}\Big)\\ &= \boxed{\frac{\ln 2}{3}}.\end{aligned}$$

9. Emily plays a game on the real line. Emily starts at the number 1 and starts with 0 points. When she is at the real number a, she chooses a real number b such that $a < b \le 100$. She then moves to b and gains $\frac{4(b-a)}{(a+b)^2}$ points. She repeats this process until she reaches the number 100. Compute the smallest possible value of c such that Emily's score is always less than c.

Answer: $\mathbf{\frac{99}{100}}$

Solution: Note that we can express $\frac{4(b-a)}{(a+b)^2} = \frac{b-a}{\left(\frac{a+b}{2}\right)^2}$. Let the sequence of points that Emily lands on be $x_0, x_1, \cdots, x_n$ where $x_0 = 1$ and $x_n = 100$. Then, Emily's score can be written as

$$\sum_{i=1}^{n} \frac{x_i - x_{i-1}}{\left(\frac{x_i+x_{i-1}}{2}\right)^2},$$

which is exactly the midpoint Riemann sum approximation for the integral

$$\int_1^{100} \frac{1}{x^2}\,\mathrm{d}x\,.$$

Moreover, since the function $f(x) = \frac{1}{x^2}$ is a concave up function, the midpoint Riemann sum approximation will always be less than the integral, and as $n \to \infty$, the Riemann sum can get infinitely close to the integral, by the definition of the integral. Thus, the smallest possible value for c is

$$\int_1^{100} \frac{1}{x^2}\,\mathrm{d}x = \left[-\frac{1}{x}\right]_1^{100} = \boxed{\frac{99}{100}}.$$

10. Compute

$$\prod_{n=1}^{\infty} \frac{\pi\arctan(n)}{2\arctan(2n)\arctan(2n-1)}.$$

Answer: $\mathbf{\sqrt[\pi]{4}}$

Solution: Let $P_N := \prod_{n=1}^{N} \frac{\pi\arctan(n)}{2\arctan(2n)\arctan(2n-1)}$ be the Nth partial product. Notice that by taking a logarithm and telescoping,

$$\ln P_N = N\ln\left(\frac{\pi}{2}\right) - \sum_{n=N+1}^{2N} \ln(\arctan(n)).$$

We compute this summation by approximating using Taylor series. Observe that if N is very large, then $1/n$ is close to 0 for $n \ge N$. Also observe that

$$\frac{\mathrm{d}}{\mathrm{d}z}\arctan\left(\frac{1}{z}\right) = \frac{-1}{z^2+1} = -1 + O(z^2).$$

and $\lim_{z\to 0^+} \arctan(1/z) = \pi/2$, so we have for large n,

$$\arctan(n) = \frac{\pi}{2} - \frac{1}{n} + O\left(\frac{1}{n^3}\right).$$

This gives for large n

$$\begin{aligned}\ln(\arctan(n)) &= \ln\left(\frac{\pi}{2} - \frac{1}{n} + O\left(\frac{1}{n^2}\right)\right) \\ &= \ln\left(\frac{\pi}{2}\right) + \ln\left(1 - \frac{2}{\pi n} + O\left(\frac{1}{n^2}\right)\right) \\ &= \ln\left(\frac{\pi}{2}\right) - \frac{2}{\pi n} + O\left(\frac{1}{n^2}\right)\end{aligned}$$

using the Taylor series for $\ln(1+x)$. Thus we compute

$$\begin{aligned}\ln P_N &= \sum_{n=N+1}^{2N}\left(\frac{2}{\pi n} + O\left(\frac{1}{n^2}\right)\right) \\ \ln P &= \lim_{N\to\infty}\int_N^{2N} \frac{2}{\pi x}\,\mathrm{d}x = \frac{2\ln 2}{\pi}\end{aligned}$$

by noting that the summation may be approximated by an integral with the approximation error vanishing in the limit $N \to \infty$. This gives us our answer of $P = 4^{1/\pi} = \boxed{\sqrt[\pi]{4}}$.