1. The *isoelectric point* of glycine is the pH at which it has zero charge. Its charge is $-\frac{1}{3}$ at pH 3.55, while its charge is $\frac{1}{2}$ at pH 9.6. Charge increases linearly with pH. What is the isoelectric point of glycine?

   Answer: 5.97

   Solution: Since charge increases linearly with pH, the pH at which glycine has zero charge is closer to 3.55 than 9.6. Precisely, its charge is $\frac{2}{5}$ of the way from 3.55 to 9.6 because its charge is $-\frac{2}{6}$ at pH 3.55 and $\frac{3}{6}$ at pH 9.6. So the isolectric point of glycine is $3.55+\frac{2}{5}(9.6-3.55) = \boxed{5.97}$.

2. The battery life on a computer decreases at a rate proportional to the display brightness. Austin starts off his day with both his battery life and brightness at 100%. Whenever his battery life (expressed as a percentage) reaches a multiple of 25, he also decreases the brightness of his display to that multiple of 25. If left at 100% brightness, the computer runs out of battery in 1 hour. Compute the amount of time, in minutes, it takes for Austin's computer to reach 0% battery using his modified scheme.

   Answer: 125

   Solution: It takes $\frac{1}{4}$ hours for Austin's display to drop 25 battery points when the brightness is at 100%, so it takes $\frac{1}{3}, \frac{1}{2}$, and $\frac{1}{1}$ hours for his battery to drop 25 percent at $75, 50$, and 25 percent brightness, respectively. Thus, our answer is $\frac{1}{4}+\frac{1}{3}+\frac{1}{2}+\frac{1}{1} = \frac{25}{12}$ hours or $\boxed{125}$ minutes.

3. Compute $\log_2 6 \cdot \log_3 72 - \log_2 9 - \log_3 8$.

   Answer: 5

   Solution: We use logarithm properties repeatedly to simplify the expression:

$$\begin{aligned}
\log_2 6 \cdot \log_3 72 - \log_2 9 - \log_3 8 &= (\log_2 3 + \log_2 2)(\log_3 2^3 + \log_3 3^2) - \log_2 3^2 - \log_3 2^3 \\
&= (\log_2 3 + 1)(3\log_3 2 + 2) - 2\log_2 3 - 3\log_3 2 \\
&= [3(\log_2 3 \cdot \log_3 2) + 2\log_2 3 + 3\log_3 2 + 2] - 2\log_2 3 - 3\log_3 2 \\
&= 3(1) + 2 = \boxed{5}.
\end{aligned}$$

4. Compute the sum of all real solutions to $4^x - 2021 \cdot 2^x + 1024 = 0$.

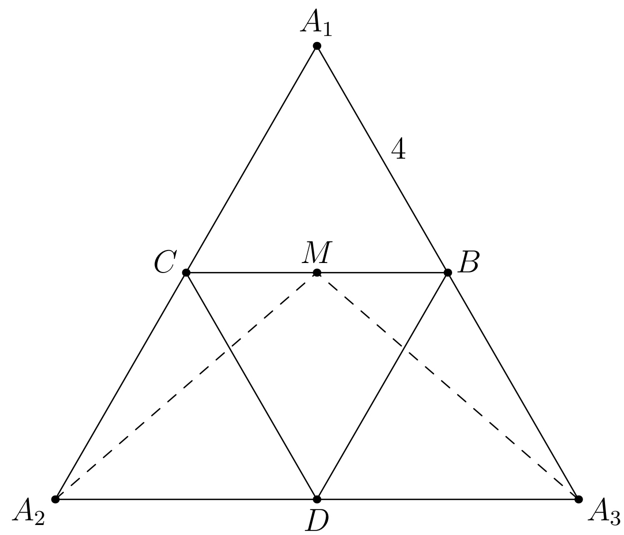
   Answer: 10

   Solution: Let $y = 2^x$, so we have $y^2 - 2021y + 1024 = 0$. Let y_1, y_2 be the roots of this equation, so $y_1y_2 = 1024$ by Vieta's. We can quickly verify that both solutions are positive reals, as $y_1 = \frac{2021+\sqrt{4080345}}{2}$ and $y_2 = \frac{2021-\sqrt{4080345}}{2}$ are both positive and real.
   Then we have $x = \log_2 y$, which is a one-to-one function, so the sum of all such x is $\log_2 y_1 + \log_2 y_2 = \log_2 y_1y_2 = \log_2 1024 = \boxed{10}$.

5. Anthony the ant is at point A of regular tetrahedron $ABCD$ with side length 4. Anthony wishes to crawl on the surface of the tetrahedron to the midpoint of $\overline{BC}$. However, he does not want to touch the interior of face $\triangle ABC$, since it is covered with lava. What is the shortest distance Anthony must travel?

Answer: $\mathbf{2\sqrt{7}}$

Solution: Let M be the midpoint of $\overline{BC}$. Unfold the tetrahedron with $\triangle BCD$ at the base. Let us call the points vertex A is split into as A_1, A_2, A_3, where A_1 forms $\triangle A_1BC$, A_2 forms $\triangle A_2CD$, A_3 forms $\triangle A_3BD$. This gives the following diagram.



Anthony cannot travel directly from A_1 to M, so he must start at either A_2 or A_3 and head in a straight line to M. By symmetry, going from A_2 to M gives the same answer as going from A_3 to M, so for the sake of simplicity, let's just consider the distance from A_2 to M. Note that equilateral triangle $\triangle A_1A_2A_3$ has side length 8, $MD = 2\sqrt{3}$, and $A_2D = 4$, and $\angle A_2DM = \frac{\pi}{2}$. We use the Pythagorean Theorem to get $\boxed{2\sqrt{7}}$.

6. Three distinct integers are chosen uniformly at random from the set

$$\{2021, 2022, 2023, 2024, 2025, 2026, 2027, 2028, 2029, 2030\}.$$

Compute the probability that their arithmetic mean is an integer.

Answer: $\mathbf{\frac{7}{20}}$

Solution: Note that this is the same as choosing 3 distinct integers whose sum is divisible by 3. Thus we can represent the set with their remainders modulo 3. There are three 0's, three 1's, and four 2's. The only ways we can pick 3 distinct integers that sum to a multiple of 3 are if we choose all the same remainders, or all different. For all 0's, the probability is $\frac{1}{\binom{10}{3}}$; for all 1's, it is $\frac{1}{\binom{10}{3}}$; for all 2's, it is $\frac{4}{\binom{10}{3}}$; and for all different, the probability is $\frac{3 \cdot 3 \cdot 4}{\binom{10}{3}}$. Thus, the total probability is $\frac{42}{\binom{10}{3}} = \boxed{\frac{7}{20}}$.

7. Ditty can bench 80 pounds today. Every week, the amount he benches increases by the largest prime factor of the weight he benched in the previous week. For example, since he started benching 80 pounds, next week he would bench 85 pounds. What is the minimum number of weeks from today it takes for Ditty to bench at least 2021 pounds?

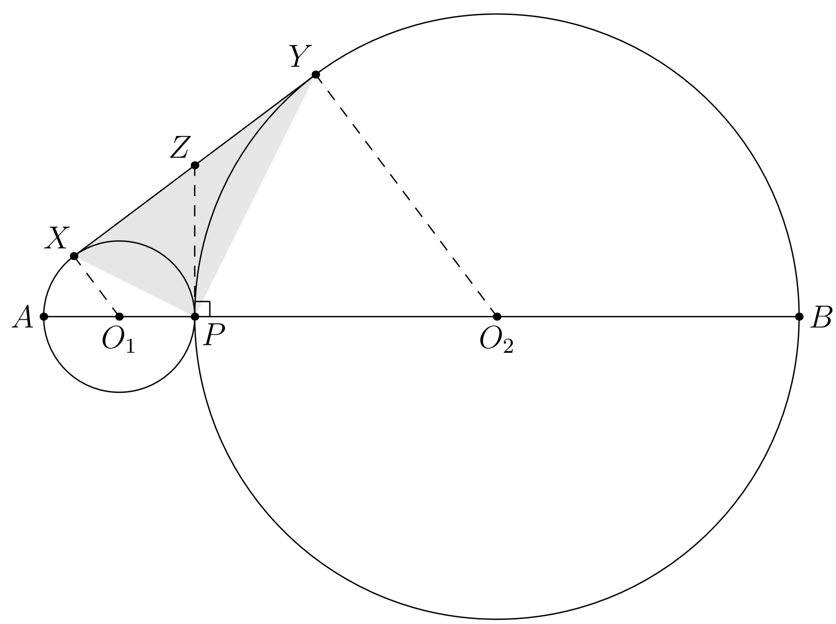
Answer: 69

Solution: After 1 week Ditty benches $85 = 5 \cdot 17$ pounds. Now, notice that the amount he benches goes up by 17 each week until he hits $17 \cdot 19$ pounds. Now the amount he benches goes up by 19 each week until hitting $19 \cdot 23$ pounds. At this point, we notice a pattern that when Ditty hits an amount that is the product of two consecutive primes, the amount he benches begins to go up by the larger prime. We can envision the progression after the first week by expressing the amount benched as the product of two integers, then increasing one of those integers every week until we hit the next prime. Since $2021 = 43 \cdot 47$, to get from $5 \cdot 17$ takes $43 - 5 + 47 - 17 = 68$, so adding the first week, the amount he benches will increase by a total of $\boxed{69}$ times.

8. Let $\overline{AB}$ be a line segment with length 10. Let P be a point on this segment with $AP = 2$. Let ω_1 and ω_2 be the circles with diameters $\overline{AP}$ and $\overline{PB}$, respectively. Let $\overleftrightarrow{XY}$ be a line externally tangent to ω_1 and ω_2 at distinct points X and Y, respectively. Compute the area of $\triangle XPY$.

Answer: $\mathbf{\frac{16}{5}}$

Solution: Let Z be the intersection of the line through P perpendicular to segment $\overline{AB}$ with segment $\overline{XY}$, and let O_1 and O_2 be the centers of circles ω_1 and ω_2. This gives the following diagram.



By equal tangents, $ZX = ZP = ZY$, so Z is the midpoint of segment $\overline{XY}$. We can calculate $XY = 4$ after dropping the altitude from O_1 to O_2Y and using the Pythagorean theorem. Since $ZX = ZP = ZY = 2$,

$$[XZP] = \frac{2^2}{2^2+1^2}[XZPO_1] = \frac{8}{5} \quad \text{and} \quad [YZP] = \frac{2^2}{2^2+4^2}[YZPO_2] = \frac{8}{5}.$$

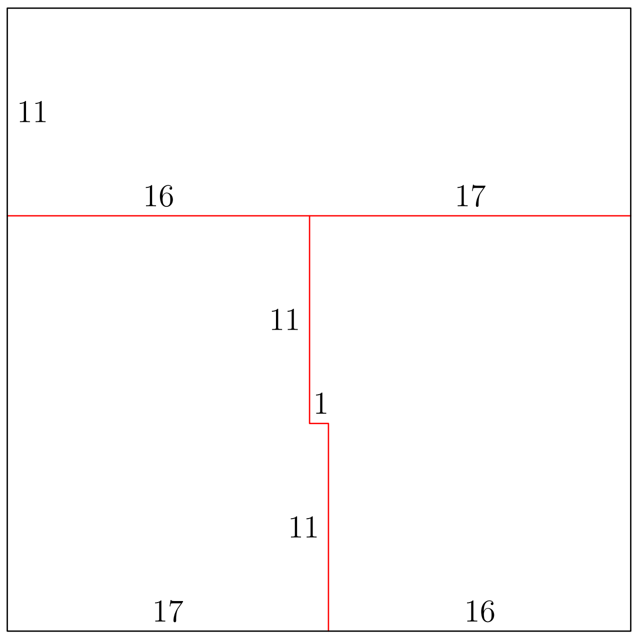
Therefore, $[XPY] = [XZP] + [YZP] = \boxed{\frac{16}{5}}$.

9. Druv has a 33×33 grid of unit squares, and he wants to color each unit square with exactly one of three distinct colors such that he uses all three colors and the number of unit squares with

each color is the same. However, he realizes that there are *internal sides*, or unit line segments that have exactly one unit square on each side, with these two unit squares having different colors. What is the minimum possible number of such internal sides?

Answer: 56

Solution: Create one 11×33 rectangle by making a horizontal cut 11 squares from the top. Then, make a vertical cut from the first cut, 16 units from the left side of the square. Make this cut 11 units long. Then, make a cut 1 unit to the right, then continue down to the bottom of the square. Note that the first rectangle has exactly one-third of the total area, and our cuts make the other two shapes the same area. This looks like the following.



This cut has 56 edges, so there are $\boxed{56}$ internal sides.

To see why we cannot do better, temporarily ignore the grid lines. The best we can do in this case is create a horizontal cut $\frac{1}{3}$ from the top, then make a vertical cut down from the midpoint of that cut, making the total cut length 55. However, we cannot actually get this when we put back the unit squares, as we'll need to include at least one more horizontal cut. Hence, the best we can do is 56.

10. Compute the number of nonempty subsets S of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ such that $\dfrac{\max S + \min S}{2}$ is an element of S.

Answer: 234

Solution: Note that the parity of $\min S$ and $\max S$ must be the same. Assume first that S has at least two elements. We condition on the values of $\min S$, assuming that $\min S \neq \max S$.

- $\min S = 1, 2$. Then, there are 4 possible values of $\max S$ (not equal to $\min S$) for which the condition $\frac{\max S + \min S}{2}$ being in S can be satisfied. For each of these values, we get $1 + 2^2 + 2^4 + 2^6 = 85$ possibilities for subsets, based on including or excluding integers x such that $\min S < x < \max S$ and $x \neq \frac{\max S + \min S}{2}$. Thus, we have $2(85) = 170$ possibilities for this case.

- $\min S = 3, 4$. Using the same argument as above, we get $2(1 + 2^2 + 2^4) = 42$ subsets for this case.
- $\min S = 5, 6$. Using the same argument as above, we get $2(1 + 2^2) = 10$ subsets for this case.
- $\min S = 7, 8$. We have $2(1) = 2$ subsets for this case.

Now, assume that S only has one element. In this case, $\min S = \max S$, so all 10 possibilities of S work. Therefore, the answer is $170 + 42 + 10 + 2 + 10 = \boxed{234}$.

11. Compute the sum of all prime numbers p with $p \geq 5$ such that p divides $(p+3)^{p-3} + (p+5)^{p-5}$.

Answer: 322

Solution: We see that $p = 5$ works. For $p > 5$, by Fermat's Little Theorem it follows that

$$(p+3)^{p-3} + (p+5)^{p-5} \equiv (p+3)^{-2} + (p+5)^{-4} \equiv 3^{-2} + 5^{-4} = \frac{634}{5625} \equiv 0 \pmod{p}.$$

We see that $5625^{-1} \pmod{p}$ must be nonzero while $634 \equiv 0 \pmod{p}$. The only $p \geq 5$ that satisfies these conditions is $p = 317$. Thus our answer is $p = 5 + 317 = \boxed{322}$.

12. Unit square $ABCD$ is drawn on a plane. Point O is drawn outside of $ABCD$ such that lines $\overleftrightarrow{AO}$ and $\overleftrightarrow{BO}$ are perpendicular. Square $FROG$ is drawn with F on $\overline{AB}$ such that $AF = \frac{2}{3}$, R is on $\overline{BO}$, and G is on $\overline{AO}$. Extend segment $\overline{OF}$ past $\overline{AB}$ to intersect side $\overline{CD}$ at E. Compute DE.

Answer: $\frac{1}{3}$

Solution: Extend $\overline{OA}$ past A to S so that $\overline{AS}$ is perpendicular to $\overline{SD}$, extend $\overline{SD}$ past D to U so that $\overline{DU}$ is perpendicular to $\overline{UC}$, and extend $\overline{UC}$ past C to M so that $\overline{CM}$ is perpendicular to $\overline{BM}$. Note that $SUMO$ is a square that circumscribes $ABCD$. In fact, it is a dilation of square $FROG$ with respect to point O. Thus, we see that $\overleftrightarrow{OE}$ passes through the center of square $SUMO$ and consequently square $ABCD$, so by symmetry $DE = BF = 1 - AF = 1 - \frac{2}{3} = \boxed{\frac{1}{3}}$.

13. How many ways are there to completely fill a 3×3 grid of unit squares with the letters B, M, and T, assigning exactly one of the three letters to each of the squares, such that no 2 adjacent unit squares contain the same letter? Two unit squares are adjacent if they share a side.

Answer: 246

Solution: We will solve this problem using casework on the first two rows.

<u>Case A</u>: The first row has 3 distinct letters.

Without loss of generality, assume that the first row contains B, M, T in that order. We do another set of casework on the second row.

- Case 1: The top two rows look like: $\begin{bmatrix} B & M & T \\ M & B & M \\ * & * & * \end{bmatrix}$ or $\begin{bmatrix} B & M & T \\ M & T & M \\ * & * & * \end{bmatrix}$.

  In terms of the bottom row, these two states are equivalent. Consider the left state. The third row can be TMB, TMT, BTB, BMB, or BMT, which is 5 choices. Thus, there are $2 \cdot 5 = 10$ choices for this case.

- Case 2: $\begin{bmatrix} B & M & T \\ M & * & B \\ * & * & * \end{bmatrix}$ or $\begin{bmatrix} B & M & T \\ T & * & M \\ * & * & * \end{bmatrix}$
  In terms of the bottom row, these two states are also equivalent. Consider the state on the left. The middle square must be T, and the bottom row can be BMT, TBT, TBM, or TMT, for 4 choices. Thus, there are 8 total choices for this case.

There are $3! = 6$ ways to arrange the first row, so there are $6 \cdot (10 + 8) = 108$ total arrangements for case A.

<u>Case B</u>: The first row has 2 distinct letters.

Note that there are also 6 possible ways to arrange this row. Without loss of generality, assume that the first row contains B, M, B. We do another set of cases:

- Case 1: The middle row is TBM or MBT.
  Then, there are 4 choices for third row (just like in second row of case A)
- Case 2: The middle row is MTM, MBM, or TBT.
  Here, there are 5 choices for third row (just like the third row of case 1 of case A).

So, there are $6 \cdot (2 \cdot 4 + 3 \cdot 5) = 138$ arrangements for case B.

In total, there are $108 + 138 = \boxed{246}$ arrangements.

14. Given an integer c, the sequence $a_0, a_1, a_2, \ldots$ is generated using the recurrence relation $a_0 = c$ and $a_i = a_{i-1}^i + 2021a_{i-1}$ for all $i \geq 1$. Given that $a_0 = c$, let $f(c)$ be the smallest positive integer n such that $a_n - 1$ is a multiple of 47. Compute

$$\sum_{k=1}^{46} f(k).$$

Answer: 1015

Solution: Fix a_0 for now, and we will let it vary later. We can show by a quick induction that $a_k \equiv a_0^{k!} \pmod{47}$: for $k = 0$, this is true; otherwise, $a_{k+1} = a_k^{k+1} + 2021a_k \equiv a_0^{(k+1)k!} \equiv a_0^{(k+1)!} \pmod{47}$. So we are computing the least positive integer n such that

$$a_0^{n!} \equiv 1 \pmod{47}.$$

Let m be the order of a_0 so that the above condition is equivalent to $n! \equiv 0 \pmod{m}$. So we want the least positive integer n such that $n! \equiv 0 \pmod{m}$.

We now let a_0 vary and do casework on the possible values of m; note that the order of an element must divide $47 - 1 = 46$ by Fermat's little theorem.

- We can have $m = 1$. Here $n = 1$ is the smallest possible because n must be a positive integer. This case occurs for exactly one value of a_0, namely $a_0 = 1$.
- We can have $m = 2$. Here $n = 2$ is the smallest possible. This case occurs for exactly one value of a_0, namely $a_0 = -1$.
- Otherwise, we can have $m \in \{23, 46\}$ so that $n = 23$ is the smallest possible. This case will occur for the other $46 - 2 = 44$ cases.

The total sum is $1 \cdot 1 + 1 \cdot 2 + 44 \cdot 23 = \boxed{1015}$.

15. Compute
$$\frac{\cos\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{24}\right)\cos\left(\frac{\pi}{48}\right)\cos\left(\frac{\pi}{96}\right)\cdots}{\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{16}\right)\cos\left(\frac{\pi}{32}\right)\cdots}.$$

Answer: $\frac{3}{2}$

Solution: Consider the partial product
$$\frac{\cos\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{24}\right)\cdots\cos\left(\frac{\pi}{3\cdot 2^n}\right)}{\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{8}\right)\cdots\cos\left(\frac{\pi}{2^n}\right)\cdots} = \frac{\sec\left(\frac{\pi}{4}\right)\sec\left(\frac{\pi}{8}\right)\cdots\sec\left(\frac{\pi}{2^n}\right)\cdots}{\sec\left(\frac{\pi}{12}\right)\sec\left(\frac{\pi}{24}\right)\cdots\sec\left(\frac{\pi}{3\cdot 2^n}\right)}.$$
Multiplying by $1 = \frac{12}{12} = \frac{3\cdot 4\sin\left(\frac{\pi}{2}\right)}{2\cdot 12\sin\left(\frac{\pi}{6}\right)}$ and recalling the identity $\sin(2x)\sec(x) = 2\sin(x)$ yields
$$\begin{aligned}\frac{\sec\left(\frac{\pi}{4}\right)\sec\left(\frac{\pi}{8}\right)\cdots\sec\left(\frac{\pi}{2^n}\right)}{\sec\left(\frac{\pi}{12}\right)\sec\left(\frac{\pi}{24}\right)\cdots\sec\left(\frac{\pi}{3\cdot 2^n}\right)} &= \frac{3}{2}\cdot\frac{4\sin\left(\frac{\pi}{2}\right)\sec\left(\frac{\pi}{4}\right)\sec\left(\frac{\pi}{8}\right)\cdots\sec\left(\frac{\pi}{2^n}\right)\cdots}{12\sin\left(\frac{\pi}{6}\right)\sec\left(\frac{\pi}{12}\right)\sec\left(\frac{\pi}{24}\right)\cdots\sec\left(\frac{\pi}{3\cdot 2^n}\right)}\\ &= \frac{3}{2}\cdot\frac{8\sin\left(\frac{\pi}{4}\right)\sec\left(\frac{\pi}{8}\right)\cdots\sec\left(\frac{\pi}{2^n}\right)}{24\sin\left(\frac{\pi}{12}\right)\sec\left(\frac{\pi}{24}\right)\cdots\sec\left(\frac{\pi}{3\cdot 2^n}\right)}\\ &= \cdots\\ &= \frac{3}{2}\cdot\frac{2^{n+1}\sin\left(\frac{\pi}{2^n}\right)}{3\cdot 2^{n+1}\sin\left(\frac{\pi}{3\cdot 2^n}\right)}.\end{aligned}$$
Now observe that the numerator of the fraction above is the perimeter of a 2^n-gon inscribed in a unit circle, and the denominator is the perimeter of a $3\cdot 2^n$-gon inscribed in a unit circle. As n approaches ∞ (i.e. as we take later partial products), these perimeters both approach 2π, the circumference of a circle. Hence, the partial products approach $\frac{3}{2}\cdot\frac{2\pi}{2\pi} = \boxed{\frac{3}{2}}$.

16. Sigfried is singing the ABC's 100 times straight, for some reason. It takes him 20 seconds to sing the ABC's once, and he takes a 5 second break in between songs. Normally, he sings the ABC's without messing up, but he gets fatigued when singing correctly repeatedly. For any song, if he sung the previous three songs without messing up, he has a $\frac{1}{2}$ chance of messing up and taking 30 seconds for the song instead. What is the expected number of minutes it takes for Sigfried to sing the ABC's 100 times? Round your answer to the nearest minute.

Answer: 45

Solution: For this problem, it suffices to know the expected number of times Sigfried messes up, which is around the number of streaks of songs that he sings without messing up. Let's investigate the length of streaks of the ABC's that Sigfried sings without messing up (including the song he messes up on). It is guaranteed that he sings at least 4 songs in a streak, and from there an additional song is added with $\frac{1}{2}$ probability, repeatedly. So a first estimate would be to obtain the expected number of streaks that occur throughout the 100 songs. The expected length of a streak is $4+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots = 5$, so we would expect there to be 20 streaks throughout the 100 songs. This gives us an estimate of $\frac{100\cdot 20+99\cdot 5+20\cdot 10}{60} = \frac{539}{12} \approx 44.92$ minutes. We also need to account for the possibility that a streak continues past the 100th song, but this doesn't actually change all that much: at most, it decreases the number of streaks by 1, and if we assume that there are an expected 19 streaks throughout the 100 songs, we get an estimate of $\frac{100\cdot 20+99\cdot 5+19\cdot 10}{60} = \frac{179}{4} = 44.75$ minutes. Either way, the expected number of minutes rounds to $\boxed{45}$.

17. Triangle $\triangle ABC$ has circumcenter O and orthocenter H. Let D be the foot of the altitude from A to $\overleftrightarrow{BC}$, and suppose $AD = 12$. If $BD = \frac{1}{4}BC$ and $\overleftrightarrow{OH} \parallel \overleftrightarrow{BC}$, compute AB^2.

Answer: 160

Solution: Draw point E so that $OE \perp BC$. Note that since O is the circumcenter, E is the midpoint of BC, and so AE is a median. Since H, G, and O lie on a line where G is the centroid, and $AD \parallel OE$, we know that $\triangle AHG \sim \triangle EOG$, and thus $AH = 2OE$. Since $OH \parallel BC$ and $HD \perp BC$, $HDEO$ is a rectangle, so $HD = OE = \frac{1}{2}AH$. Thus, $AH = 8, HD = 4$. Let $BD = x$. Since E is a midpoint, we know that $DE = x$ and $EC = 2x$. As O is a circumcenter, $AO = OC$, so $\sqrt{AH^2 + OH^2} = \sqrt{OE^2 + EC^2}$. Solving for x, we get $x = 4$. Finally, using the Pythagorean Theorem, $AB^2 = 12^2 + 4^2 = \boxed{160}$.

18. The equation $\sqrt[3]{\sqrt[3]{x - \frac{3}{8}} - \frac{3}{8}} = x^3 + \frac{3}{8}$ has exactly two real *positive* solutions r and s. Compute $r + s$.

Answer: $\mathbf{\frac{1+\sqrt{13}}{4}}$

Solution: Rearrange the equation as $\sqrt[3]{\sqrt[3]{x - \frac{3}{8}} - \frac{3}{8}} - \frac{3}{8} = x^3$, then take the cube root of both sides to get $\sqrt[3]{\sqrt[3]{\sqrt[3]{x - \frac{3}{8}} - \frac{3}{8}} - \frac{3}{8}} = x$. Then observe that if we let $f(x) = \sqrt[3]{x - \frac{3}{8}}$, the equation becomes $f(f(f(x))) = x$, which is satisfied by the solutions to $f(x) = x$, which becomes $\sqrt[3]{x - \frac{3}{8}} = x$. So $x^3 - x + \frac{3}{8} = 0$. We can note that one solution is $\frac{1}{2}$, reducing the equation down to the quadratic $x^2 + \frac{1}{2}x - \frac{3}{4} = 0$. This quadratic has solutions $\frac{-1\pm\sqrt{13}}{4}$, and the positive solution is $\frac{-1+\sqrt{13}}{4}$. We have found both positive real solutions, so $r + s = \frac{1}{2} + \frac{-1+\sqrt{13}}{4} = \boxed{\frac{1+\sqrt{13}}{4}}$.

19. Let a be the answer to **Problem 19**, b be the answer to **Problem 20**, and c be the answer to **Problem 21**.

Compute the real value of a such that

$$\sqrt{a(101b + 1)} - 1 = \sqrt{b(c - 1)} + 10\sqrt{(a - c)b}.$$

Answer: 2021

Solution: Note that by Cauchy-Schwarz inequality on $(1, \sqrt{b}, 10\sqrt{b})$ and $(1, \sqrt{c-1}, \sqrt{a-c})$, we can see that

$$\begin{aligned}(1 + (\sqrt{b})^2 + (10\sqrt{b})^2)(1 + (\sqrt{c-1})^2 + (\sqrt{a-c})^2) &\geq (1 \cdot 1 + \sqrt{b} \cdot \sqrt{c-1} + 10\sqrt{b} \cdot \sqrt{a-c})^2 \\ (101b + 1)a &\geq (1 + \sqrt{b(c-1)} + 10\sqrt{b(a-c)})^2 \\ \sqrt{a(101b+1)} - 1 &\geq \sqrt{b(c-1)} + 10\sqrt{(a-c)b}\end{aligned}$$

Therefore, we must satisfy the equality case which means $\frac{1}{1} = \frac{\sqrt{c-1}}{\sqrt{b}} = \frac{\sqrt{a-c}}{10\sqrt{b}}$. This translates to $a = 100b + c$ and $c = b + 1$ respectively as desired. With $b = 20$ and $c = 21$, we get $a = \boxed{2021}$.

20. Let a be the answer to **Problem 19**, b be the answer to **Problem 20**, and c be the answer to **Problem 21**.

For some triangle $\triangle ABC$, let ω and ω_A be the incircle and A-excircle with centers I and I_A, respectively. Suppose $\overleftrightarrow{AC}$ is tangent to ω and ω_A at E and E', respectively, and $\overleftrightarrow{AB}$ is tangent to ω and ω_A at F and F', respectively. Furthermore, let P and Q be the intersections of $\overleftrightarrow{BI}$ with $\overleftrightarrow{EF}$ and $\overleftrightarrow{CI}$ with $\overleftrightarrow{EF}$, respectively, and let P' and Q' be the intersections of $\overleftrightarrow{BI_A}$ with $\overleftrightarrow{E'F'}$ and $\overleftrightarrow{CI_A}$ with $\overleftrightarrow{E'F'}$, respectively. Given that the circumradius of $\triangle ABC$ is a, compute the maximum integer value of BC such that the area $[PQP'Q']$ is less than or equal to 1.

Answer: 20

Solution: Note that this is an Iran Lemma problem on incircles and excircles. First, allow D and D' to be the tangency points of ω and ω_A to $\overline{BC}$ respectively. We can prove that $PQP'Q'$ is actually a rectangle by directed angle chasing. We see that

$$\measuredangle BIQ = \measuredangle BIC = \frac{\pi}{2} + \frac{\measuredangle BAC}{2} = \measuredangle AFE = \measuredangle BFQ,$$

meaning $BIQF$ is cyclic. Therefore, $\measuredangle BQC = \measuredangle BQI = \measuredangle BFI = \frac{\pi}{2}$. We can use the above to show $\measuredangle BQ'C = \frac{\pi}{2}$ as well, namely

$$\measuredangle BI_AQ' = \measuredangle BI_AC = \measuredangle BIC = \measuredangle AFE = \measuredangle AF'E' = \measuredangle BFQ',$$

meaning $BI_AQ'F'$ is cyclic. With this we achieve $\measuredangle BQ'C = \measuredangle BQ'I_A = \measuredangle BF'I_A = \frac{\pi}{2}$. By symmetry, we can obtain that $\measuredangle BPC = \measuredangle BP'C = \frac{\pi}{2}$ as well, meaning $BP'Q'CPQ$ is cyclic with diameter $\overline{BC}$. Allowing M to be the midpoint of $\overline{BC}$ and hence the circumcenter of $\odot(BP'Q'CPQ)$, we get

$$\measuredangle BMP = 2\measuredangle BQP = 2\measuredangle BQF = 2\measuredangle BIF = \measuredangle DIF = \measuredangle DBF = \measuredangle CBA.$$

Similarly,

$$\measuredangle BMP' = 2\measuredangle BQ'P' = 2\measuredangle BQ'F' = 2\measuredangle BI_AF' = \measuredangle D'I_AF' = \measuredangle D'BF' = \measuredangle CBA.$$

Therefore, because $\measuredangle BMP = \measuredangle CBA = \measuredangle BMP'$, P, M, and P' must be collinear which means $\overline{PP'}$ is a diameter of $\odot(BP'Q'CPQ)$. By symmetry we know $\overline{QQ'}$ is a diameter as well, which means $PQP'Q'$ is indeed a rectangle. To find the area, we can first see that

$$\measuredangle PMQ = 2\measuredangle PBQ = 2\measuredangle IBQ = 2\measuredangle IFQ = 2\measuredangle IFE = 2\measuredangle IAE = \measuredangle BAC.$$

Therefore, the area $[PQP'Q'] = 4 \cdot \frac{1}{2} PM \cdot QM \cdot \sin \angle PMQ = \frac{BC^2}{2} \cdot \sin \angle BAC = \frac{BC^2}{2} \cdot \frac{BC}{2a} = \frac{BC^3}{4a}$. Since we want BC as the maximal integer value that makes this area less than or equal to 1, we get that our answer is $BC = \left\lfloor \sqrt[3]{4a} \right\rfloor$. Given that $a = 2021$ from Problem 19, we get: $\left\lfloor \sqrt[3]{8084} \right\rfloor = \boxed{20}$.

21. Let a be the answer to **Problem 19**, b be the answer to **Problem 20**, and c be the answer to **Problem 21**.

Let c be a positive integer such that $\gcd(b, c) = 1$. From each ordered pair (x, y) such that x and y are both integers, we draw two lines through that point in the x-y plane, one with slope $\frac{b}{c}$ and one with slope $-\frac{c}{b}$. Given that the number of intersections of these lines in $[0, 1)^2$ is a square number, what is the smallest possible value of c? Note that $[0, 1)^2$ refers to all points (x, y) such that $0 \le x < 1$ and $0 \le y < 1$.

Answer: 21

Solution: First, note that every line drawn of slope $\frac{b}{c}$ can be written in the form $-bx+cy=k$ where k is an integer. Furthermore, by Bezout's Theorem, because $\gcd(b,c)=1$, for every integer k there exists a lattice point (x,y) such that $-bx+cy=k$, which means that the set of lines drawn of slope $\frac{b}{c}$ is exactly those of the form $-bx+cy=k$ for all integers k. A similar argument can be made to conclude that the set of lines drawn of slope $-\frac{c}{b}$ is exactly lines of the form $cx+by=k$ for all integers k. Then, because lines of slope $-\frac{c}{b}$ are perpendicular to lines of the slope $\frac{b}{c}$, the constructed lines form a lattice coordinate system p,q where $p=-bx+cy$ and $q=cx+by$.

We claim that if two lines $-bx+cy=p'$ and $cx+by=q'$ where p',q' are integers intersect at (x',y'), then x' is an integer if and only if y' is an integer. Solving the two equations produces the result $(x',y')=\left(\frac{-bp'+cq'}{b^2+c^2},\frac{cp'+bq'}{b^2+c^2}\right)$. Given x' is an integer,

$$bx'+p'=b\cdot\frac{-bp'+cq'}{b^2+c^2}+p'=\frac{c^2p'+bcq'}{b^2+c^2}=\frac{c(cp'+bq')}{b^2+c^2}.$$

must also be an integer. Since $\gcd(b,c)=1$, $\gcd(c,b^2+c^2)=\gcd(c,b^2)=1$, meaning $cp'+bq'$ must be divisible by b^2+c^2 and thus, y' must be an integer as well. A similar argument can be made to prove the reverse case.

Then, note that the square $(0,0),(1,0),(1,1),(0,1)$ is represented by $(0,0),(-b,c),(-b+c,c+b),(c,b)$ in the p,q-coordinate system. We can note that the area in the p,q-coordinate system is simply that of a square of side length $\sqrt{b^2+c^2}$, meaning its area is b^2+c^2. We can then use Pick's Theorem in the p,q-coordinate system to compute the number of interior points in the square $[0,1)^2$. We have

$$I+\frac{B}{2}-1=b^2+c^2$$

where I is the number of interior points and B is the number of boundary points. However, we have proven that if a lattice point in p,q lies on the edge of the square, it must be one of the vertices, or otherwise, one of the x,y-coordinates will not be an integer. Therefore, we have $B=4$ for the four vertices, meaning $I=b^2+c^2+1-\frac{4}{2}=b^2+c^2-1$. This means the drawn lines intersect strictly within the region $(0,1)^2$ a total of b^2+c^2-1 times. Then noting that $(0,0)\in[0,1)^2$ in x,y corresponds to the p,q lattice point $(0,0)$, the total number of intersections in the region $[0,1)^2$ is thus $(b^2+c^2-1)+1=b^2+c^2$.

We then conclude that the answer must be the leg of a Pythagorean Triple. From the other questions, the answer must be one more than the other leg, greatly eliminating the possible answers. We find that only $c=\boxed{21}$ satisfies the conditions for all three problems.

22. In $\triangle ABC$, let D and E be points on the angle bisector of $\angle BAC$ such that $\angle ABD=\angle ACE=90^\circ$. Furthermore, let F be the intersection of $\overleftrightarrow{AE}$ and $\overleftrightarrow{BC}$, and let O be the circumcenter of $\triangle AFC$. If $\frac{AB}{AC}=\frac{3}{4}$, $AE=40$, and $\overline{BD}$ bisects $\overline{EF}$, compute the perpendicular distance from A to $\overleftrightarrow{OF}$.

Answer: $\mathbf{\frac{10\sqrt{21}}{3}}$

Solution: The key claim here is that F is the circumcenter of $\triangle ACE$. This follows from the given conditions and the fact that $\triangle ABD\sim\triangle ACE$, since F is then the midpoint of hypotenuse $\overline{AE}$ in right triangle $\triangle ACE$.

Because $\overline{AC}$ is the radical axis of $\triangle ACE$ and $\triangle ACF$, we must have that $\overline{OF}$ perpendicularly bisects $\overline{AC}$, meaning it suffices to calculate AC. This can be done via trig, but a clean synthetic way is to note that $\angle BCA = \angle FCA = \angle CAF = \angle FAB$, so $\overline{BA}$ is tangent to $\triangle AFC$. From $AE = 40$ and F being its midpoint, we see that $FE = FC = 20$, so the angle bisector theorem gives $BF = 15$. Then by Power of a Point,

$$AB^2 = BF \cdot BC \implies AB = 5\sqrt{21} \implies AC = \frac{20\sqrt{21}}{3}$$

The answer is half that, or $\boxed{\frac{10\sqrt{21}}{3}}$.

23. Alireza is currently standing at the point $(0, 0)$ in the x-y plane. At any given time, Alireza can move from the point (x, y) to the point $(x + 1, y)$ or the point $(x, y + 1)$. However, he cannot move to any point of the form (x, y) where $y \equiv 2x \pmod 5$. Let p_k be the number of paths Alireza can take starting from the point $(0, 0)$ to the point $(k + 1, 2k + 1)$. Evaluate the sum

$$\sum_{k=1}^{\infty} \frac{p_k}{5^k}.$$

Answer: $\frac{13}{9}$

Solution: The key idea is that for any integer a, once Alireza travels below the line $y = 2x+5a$, he cannot move above the line from that point onward. This can be noted since there is a lattice point for every line $x = k$ for integer k, creating a barrier that cannot be passed once the line is traveled below. So, in order to get to the point $(k + 1, 2k + 1)$, Alireza must stay below the line $y = 2x + 5$, and he is also already bounded above the line $y = 2x - 5$.

This means that there are many choke points where Alireza must pass through, which we can apply recursion to. In particular, he must pass through either of the pair of points $A_k = (k + 1, 2k - 1)$ and $B_k = (k - 1, 2k + 1)$ for all positive integer k. Let a_k and b_k be the number of ways to get to A_k and B_k, respectively. Then using path traversal counting, we get $a_1 = 3$, $b_1 = 1$, $a_k = 2a_{k-1} + b_{k-1}$, and $b_k = 2b_{k-1}$. Then it is apparent that $b_k = 2^{k-1}$, and using induction or recursion solving techniques it can be found that $a_k = 2^{k-2}(k + 5)$. The number of ways to get to the point $(k + 1, 2k + 1)$ is $p_k = a_k + b_k = 2^{k-2}(k + 7)$.

There are a few ways to compute

$$\sum_{k=1}^{\infty} \frac{2^{k-2}(k + 7)}{5^k} = \frac{1}{4}\left(\frac{14}{3} + \sum_{k=1}^{\infty} k\left(\frac{2}{5}\right)^k\right)$$

which is an arithmetico-geometric sequence. Let $S = \sum_{k=1}^{\infty} k\left(\frac{2}{5}\right)^k$. Then

$$\begin{aligned}
\frac{5}{2}S &= \sum_{k=1}^{\infty} k\left(\frac{2}{5}\right)^{k-1} \\
&= \sum_{k=0}^{\infty}(k + 1)\left(\frac{2}{5}\right)^k \\
&= \frac{5}{3} + \sum_{k=0}^{\infty} k\left(\frac{2}{5}\right)^k \\
&= \frac{5}{3} + S
\end{aligned}$$

and solving gives $S = \frac{10}{9}$. Finally, the desired sum is $\frac{1}{4}\left(\frac{14}{3} + \frac{10}{9}\right) = \boxed{\frac{13}{9}}$.

24. Suppose that a, b, c, and p are positive integers such that p is a prime number and

$$a^2 + b^2 + c^2 = ab + bc + ca + 2021p.$$

Compute the least possible value of $\max(a, b, c)$.

Answer: 330

Solution:

Let $N = 2021p$. We first use the substitution $x = a-b$ and $y = a-c$. This converts the equation to

$$x^2 - xy + y^2 = N,$$

which corresponds to the norm of Eisenstein integers.

The Eisenstein integers are complex numbers of the form $x + y\omega$, where x and y are integers and $\omega = e^{2i\pi/3}$ is a third root of unity. They form a commutative ring, and the norm of an Eisenstein integer is $N(x + y\omega) = x^2 - xy + y^2$. The norm is a multiplicative function: any pair of Eisenstein integers α, β satisfies the property $N(\alpha)N(\beta) = N(\alpha\beta)$. Furthermore, every Eisenstein integer can be factored into finitely many Eisenstein primes, which themselves cannot be factored up to units. So, it would suffice to know all possible norms of Eisenstein primes.

Known theory states that there are two types of Eisenstein primes. The first type is that any prime p in the integers that is 2 mod 3 is an Eisenstein prime, with norm p^2. The second type is that any prime p in the integers that is 1 mod 3 or equal to 3 is equal to $x^2 - xy + y^2$ for some integers x, y, and can be factored as $(x + y\omega)(x + y\omega^2)$, where the factors $x + y\omega$ and $x + y\omega^2$ are Eisenstein primes with norm p. Since all Eisenstein integers can be factored into Eisenstein primes, the norm of any Eisenstein integer must have each prime which is 2 mod 3 be raised to an even power. We conclude that there exist integers x, y such that $x^2 - xy + y^2 = N$ if and only if for all primes p dividing N that are 2 mod 3, $v_p(N)$ is even.

Back to the original problem: we know that $2021p = x^2 - xy + y^2$ is the norm of $x + \omega y$. We have the prime factorization $2021 = 43 \cdot 47$, and from the above facts, $v_{47}(2021p)$ is even, so we deduce that p must be equal to 47. Since 47 is an Eisenstein prime, the only way to factor out 47 from $x + \omega y$ is to write it as $47(x' + \omega y')$, where we let $x' = \frac{x}{47}$ and $y' = \frac{y}{47}$ and we must have that x, y are both divisible by 47. The norm of $x' + \omega y'$ is then 43, and we now easily find solutions to $x'^2 - x'y' + y'^2 = 43$: $(x', y') = \pm(6, 7), \pm(1, 7), \pm(-1, 6)$ and permutations thereof, and the solutions for (x, y) are obtained by multiplying by 47.

For all of these solutions for (x, y), which is $(a - b, a - c)$, the greatest difference between a, b, and c is $47 \cdot 7 = 329$. Since a, b, and c are positive integers, the least possible value of $\min(a, b, c)$ is 1, so the least possible value of $\max(a, b, c)$ is $1 + 329 = \boxed{330}$. It turns out that 330 is also sufficient: two n-tuples that satisfy the condition are $(330, 48, 1, 47)$ and $(330, 283, 1, 47)$ (these are the only two, up to permutation).

25. For any $p, q \in \mathbb{N}$, we can express $\frac{p}{q}$ as the base 10 decimal $x_1x_2 \ldots x_\ell.x_{\ell+1} \ldots x_a\overline{y_1y_2 \ldots y_b}$, with the digits $y_1, \ldots y_b$ repeating. In other words, $\frac{p}{q}$ can be expressed with integer part $x_1x_2 \ldots x_\ell$ and decimal part $0.x_{\ell+1} \ldots x_a\overline{y_1y_2 \cdots y_b}$. Given that $\frac{p}{q} = \frac{(2021)^{2021}}{2021!}$, estimate the minimum

value of a. If E is the exact answer to this question and A is your answer, your score is given by $\max\left(0, \left\lfloor 25 - \frac{1}{10}|E-A| \right\rfloor\right)$.

Answer: 2889

Solution: The minimal value of a is given by the number of digits to the left of the decimal point plus the minimal number of digits to the right of the decimal point. The number of digits to the left of the decimal point can be estimated by $\left\lfloor \log_{10} \frac{(2021)^{2021}}{2021!} \right\rfloor + 1 = 876$. The number of digits to the right of the decimal point is the number of factors of 2 in 2021!, which can be calculated exactly as 2013. Thus, $876 + 2013 = \boxed{2889}$. A crude approximation of $\log_{10} \frac{(2021)^{2021}}{2021!}$ can be made by noting $\frac{(2021)^{2021}}{2021!} \approx e^{2021}$, and by approximating $e = 3$ and $\log_{10} e = 0.45$, we get an answer of 2922, which is 21 points.

26. Kailey starts with the number 0, and she has a fair coin with sides labeled 1 and 2. She repeatedly flips the coin, and adds the result to her number. She stops when her number is a positive perfect square. What is the expected value of Kailey's number when she stops? If E is your estimate and A is the correct answer, you will receive $\left\lfloor 25e^{-\frac{5|E-A|}{2}} \right\rfloor$ points.

Answer: 3.59036568873322

Solution: Consider the probability p_n that Kailey gets to the number n without the restriction of stopping at a perfect square. The recursion for this is $p_n = \frac{1}{2}p_{n-1} + \frac{1}{2}p_{n-2}$. Note that the sequence just takes the average repeatedly, and consecutive numbers get closer and closer together quite quickly.

Now, let's include the perfect square stopping condition. With this recursion, we can calculate the probability that Kailey lands on a given perfect square, so the expected value may be computed. We also need to include the priority condition $p_{k^2} = 0$ for all positive integer k. Then the probability that the ending number is k^2 is $\frac{1}{2}p_{k^2-1} + \frac{1}{2}p_{k^2-2}$. After each perfect square, the probabilities of reaching future numbers is approximately divided by a factor of 3: to see this, note that after the terms $a, 0$, the sequence will converge to $\frac{a}{3}$. So, after computing the first few probabilities, like for 1, 4, and 9, we can approximate the rest of the probabilities by a geometric sequence. Finally, we sum up the expected value portions by ways of a geometric-like sum to obtain our approximate answer.

Here's this method in action: we compute $p_0, p_1, p_2, \ldots, p_9$ to be $1, 0, \frac{1}{2}, \frac{1}{4}, 0, \frac{1}{8}, \frac{1}{16}, \frac{3}{32}, \frac{5}{64}, 0$. So the probability of ending at 1 is $\frac{1}{2}$, the probability of ending at 4 is $\frac{3}{8}$, and the probability of ending at 9 is $\frac{11}{128}$. For subsequent squares, the probability is approximately divided by 3 every time, so the rest of the expected value is the sum $\sum_{k=4}^{\infty} \left(\frac{1}{3}\right)^{k-3} \cdot \frac{11}{128} \cdot k^2 = \frac{297}{128} \sum_{k=4}^{\infty} \frac{k^2}{3^k}$. Let

$S = \sum_{k=4}^{\infty} \frac{k^2}{3^k}$; then

$$\begin{aligned}
3S &= \sum_{k=4}^{\infty} \frac{k^2}{3^{k-1}} \\
&= \sum_{k=3}^{\infty} \frac{(k+1)^2}{3^k} \\
&= \frac{16}{27} + S + \sum_{k=4}^{\infty} \frac{2k+1}{3^k} \\
&= \frac{11}{18} + S + 2\sum_{k=4}^{\infty} \frac{k}{3^k}.
\end{aligned}$$

Let $T = \sum_{k=4}^{\infty} \frac{k}{3^k}$; then

$$\begin{aligned}
3T &= \sum_{k=4}^{\infty} \frac{k}{3^{k-1}} \\
&= \sum_{k=3}^{\infty} \frac{k+1}{3^k} \\
&= \frac{4}{27} + T + \sum_{k=4}^{\infty} \frac{1}{3^k} \\
&= \frac{1}{6} + T
\end{aligned}$$

and solving the equation $3T = T + \frac{1}{6}$ gives $T = \frac{1}{12}$. Substituting back gives $3S = \frac{16}{27} + S + 2 \cdot \frac{1}{12}$, and solving for S gives $S = \frac{7}{18}$. Finally, the total approximate expected value is $1 \cdot \frac{1}{2} + 4 \cdot \frac{3}{8} + 9 \cdot \frac{11}{128} + \frac{297}{128} \cdot \frac{7}{18} = \frac{941}{256} = 3.67578125$. This approximation would score 20 points, but extending this method to further squares will yield even more accurate results. Indeed, calculating values for k up to 4 and using our approximation for $k \geq 5$ gives us $1 \cdot \frac{1}{2} + 4 \cdot \frac{3}{8} + 9 \cdot \frac{11}{128} + \frac{7}{18} \cdot \frac{17415}{8192} = \frac{58985}{16384} \approx 3.60015689$, giving us $\lfloor 24.4 \rfloor$ points (which is as many points as possible).
The Python code used to obtain the answer to 14 decimal places is as follows:

```
a = [1,0]
for i in range(2,9000000):
if int(math.sqrt(i))**2 - i == 0:
a.append(0)
else:
a.append(a[-1]*0.5+a[-2]*0.5)
print(sum([i**2 * (0.5*a[i**2-1] + 0.5*a[i**2-2]) for i in range(1000)]))
# outputs 3.5903656887332227
print(sum([i**2 * (0.5*a[i**2-1] + 0.5*a[i**2-2]) for i in range(3000)]))
# outputs 3.5903656887332227
```

27. Let $S = \{1, 2, 2^2, 2^3, \ldots, 2^{2021}\}$. Compute the difference between the number of even digits and the number of odd digits across all numbers in S (written as integers in base 10 with no leading zeros). If E is the exact answer to this question and A is your answer, your score is given by $\max\left(0, \left\lfloor 25 - \frac{1}{2 \cdot 10^8}|E - A|^4 \right\rfloor\right)$.

Answer: 1776

Solution: Note that most elements of S will end in an even number, so intuitively there will be more even digits than odd digits. One way to approximate this is to assume that the digits besides the first few and last few digits are effectively random. If we ignore the first few digits, each units digit besides 1 is even, and hence using only this assumption there will roughly be 2020 more even digits than odd digits, giving us 7 points. Now we take the first few digits into account. The first few digits follow Benford's Law, which we can use to approximate our result. Using only one leading digit and one last digit gives us an answer of 1580, which already gives us 17 points. We can improve our estimation by taking more digits into account. Using the first 2 digits gives 1600, and using the first three digits gives 1608, giving us 20 and 21 points, respectively. The exact answer turns out to be $\boxed{1776}$.

The Python code used to obtain the exact answer is as follows:

```
a = "".join([str(1 << i) for i in range(2022)])
b = [a.count(str(i)) for i in range(10)]
print(sum([b[i] * (-1 if i % 2 else 1) for i in range(10)]))
# outputs 1776
```